Chapter 4

BIVARIATE RESIDUAL LIFE DISTRIBUTION

4.1. Introduction

While reviewing the literature on characterization of life distributions in Section 1.3 reference was made to the notion of residual life distribution (RLD) and it was pointed out there that the basis of many characterizations can be traced to the form of the RLD. However, this point has only been recognized very recently. A comparison of the RLD and the parent distribution is informative in the study of the reliability characteristics. Some results using this approach in the univariate case have been reported in Gupta and Kirmani (1990). Since this concept does not appear to have been introduced in the bivariate case, the primary concern of the present chapter is to define bivariate RLD. The failure rate (MRL) determines a distribution uniquely and therefore, it follows that the failure rate (MRL) of the RLD will enable us to identify the form of RLD. In situations where the failure rate (MRL) of the basic distribution has the same form as the corresponding characteristic of the RLD, we can conclude that the parent distribution
is form-invariant with respect to the construction of the RLD. This argument will be used in the forthcoming discussions to identify a certain class of distributions that possess the closure property just described. We also infer the structure of the basic life pattern from that of the distribution of residual life.

4.2. Definition and Examples

As in the previous chapter we suppose that \( X = (X_1, X_2) \) is a random vector admitting absolutely continuous survival function \( R(x_1, x_2) \) with respect to Lebesgue measure in the support of \( R_2^+ \). For \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) in \( R_2^+ \), the bivariate RLD is defined by the survival function

\[
G(y; x) = P[X > x + y | X > x]
\]

for all \( y \in R_2^+ \) and those values of \( X \) for which \( R(x) > 0 \). In the above definition the ordering \( X > x \) implies \( X_1 > x_1 \) and \( X_2 > x_2 \).
Examples

Here, we give examples of some bivariate distributions and their residual life distributions.

1. When \( X \) follows Gumbel's exponential distribution with survival function (3.2), the bivariate RLD of \( X \) is given by

\[
G(y; x) = e^{-(\alpha_1 + \Theta x_2)y_1 - (\alpha_2 + \Theta x_1)y_2 - \Theta y_1 y_2} \quad y_1, y_2 > 0,
\]

which is again Gumbel's exponential model with parameters \((\alpha_1 + \Theta x_2), (\alpha_2 + \Theta x_1)\) and \(\Theta\).

2. For the Marshall–Olkin distribution specified by

\[
R(x_1, x_2) = e^{-a_1 x_1 - a_2 x_2 - a_{12} \max(x_1, x_2)}, \quad x_1, x_2 > 0; \quad a_1, a_2 > 0, \quad a_{12} \geq 0,
\]

the RLD is of the form,

\[
R(y; x) = e^{-a_1 y_1 - a_2 y_2 - a_{12} [\max(x_1 + y_1, x_2 + y_2) + \max(x_1, x_2)]}
\]

where \( Y(x) = \{y | Y^{-1}(y)(x) \} \) is the random variable with survival function (4.1).
3. In the case of bivariate Pareto distribution with survival function (3.3), the RLD is given by

\[ G(y; x) = \left[ 1 + \frac{(a_1 + bx_2)y_1 + (a_2 + bx_1)y_2 + by_1y_2}{(1 + a_1x_1 + a_2x_2 + bx_1x_2)} \right]^{-c} \]

which is again bivariate Pareto.

4. When the distribution of X is bivariate finite range given in (3.4), the RLD of X follows again finite range, specified by

\[ G(y; x) = \left[ 1 - \frac{(p_1 - qx_2)y_1 + (p_2 - qx_1)y_2 - qy_1y_2}{(1 - p_1x_1 - p_2x_2 + qx_1x_2)} \right]^d \]

4.3. Properties

1. \( E Y(x) = r(x) \), (4.2)

where, \( Y(x) = (Y_1(x), Y_2(x)) \) is the random variable with survival function (4.1).

Proof:

\[ E Y_1(x) = \int_0^\infty \int_0^\infty Y_1[R(x_1, x_2)]^{-1} \frac{\partial^2 R}{\partial y_1 \partial y_2} dy_1 dy_2 \]
Thus, the mean of the RLD, is the MRL vector we have encountered in Chapter 1.

Further, the marginal survival functions of (2.1) are given by

$$ G_i(\gamma_i; x) = \frac{R(x_1 + \gamma_i, x_2)}{R(x_1, x_2)} , \quad \gamma_i > 0$$

(4.3)

Similarly, $k_{ij}(\gamma_1, \gamma_2) = k_{ij}(x_1 + \gamma_1, x_2 + \gamma_2)$.

Since, the MRL determines the corresponding distribution uniquely, the functional forms of $E Y(x)$ characterize the probability law of $X$.

2. In the following discussion, we denote by $h(x) = (h_1(x), h_2(x))$ and $k(x) = (k_1(x), k_2(x))$, the vector valued failure rates of $X$ and $Y$ respectively and
by \( r(x) = (r_1(x), r_2(x)) \) and \( s(x) = (s_1(x), s_2(x)) \) the corresponding MRL's, we then have

\[
k_i(y) = h_i(x+y) \quad (4.4)
\]

and

\[
s_i(y) = r_i(x+y), \text{ for } i=1,2. \quad (4.5)
\]

**Proof:**

For \( i=1, \)

\[
k_1(y_1, y_2) = \frac{\partial \log G}{\partial y_1} = -\frac{\partial \log R(x_1+y_1, x_2+y_2)}{\partial y_1}
\]

\[
= h_1(x_1+y_1, x_2+y_2).
\]

Similarly, \( k_2(y_1, y_2) = h_2(x_1+y_1, x_2+y_2). \)

The MRL, for \( i=1, \)

\[
s_1(y_1, y_2) = G(y;x)^{-1} \int_{Y_1} G(y;x)dy.
\]
\[ [R(x+y)]^{-1} \int_{y_1}^{\infty} R(x_1+t, x_2+y_2) \, dt \]

On similar lines,
\[ s_2(y_1, y_2) = r_2(x_1+y_1, x_2+y_2). \]

From the above result, when \( r_i(x) \) is linear in \( x_i \), the MRL corresponding to \( Y \), \( s_i(x) \), is also linear in \( y_i \). This shows that \( X \) follows the distributions defined in the equations (3.2), (3.3) and (3.4), if and only if the distribution of \( Y(x) \) is of the same form.

3. The distribution of \( X \) is bivariate IFR(1) (DFR(1)) if and only if

\[ k_i(y) \geq (\leq) h_i(y), \quad (4.6) \]

for all \( y_i > 0 \), \( i = 1, 2 \).
Proof:

A bivariate random vector $X$ is said to have IFR(1) property if and only if

$$h_i(x_1+y_1, x_2+y_2) \geq (\leq) h_i(y_1, y_2)$$

for all $x_i, y_i > 0$, $i = 1, 2$.

Therefore, from (4.4),

$$X$$ is IFR(1) (DFR(1)) if and only if

$$k(y) \geq (\leq) h_i(y).$$

4. $X$ is bivariate DMRL (1) (IMRL(1)) if and only if

$$s_i(y) \leq (\geq) r_i(y)$$

for all $y_i > 0$, $i = 1, 2$.

Proof:

From Zahedi (1985),

$$X$$ is DMRL(1) (IMRL(1)) if and only if

$$r_i(x_1+y_1, x_2+y_2) \leq (\geq) r_i(y_1, y_2)$$

for all $x_i, y_i > 0$, $i = 1, 2$.

or

$$s_i(y) \leq (\geq) r_i(y).$$
5. \( X \) is bivariate NBU(VS) (NWU(VS)) if and only if
\[
G(y) \leq (\geq) R(y)
\] (4.8)
for all \( y_1, y_2 > 0 \).

Proof:

From the definition (Buchanan and Singpurwalla (1977)), \( X \) is NBU(VS) (NWU(VS)) if
\[
R(x_1+y_1, x_2+y_2) \leq (\geq) R(x_1, x_2)R(y_1, y_2)
\]
for all \( x_i, y_i > 0, i = 1, 2 \), or
\[
G(y_1, y_2) \leq (\geq) R(y_1, y_2).
\] (4.9)

6. For \( i = 1, 2 \), \( h_i(y) \leq k_i(y) \) if and only if \( \frac{R(y)}{G(y)} \) is non-decreasing in \( y_i \), for all \( y_1, y_2 > 0 \).

Proof:

\[
h_i(y_1, y_2) - k_i(y_1, y_2) \leq 0
\]
\[
\iff - \frac{\partial \log \frac{R(y)}{G(y)}}{\partial y_i} \leq 0
\]
\[
\iff \frac{\partial \log \frac{R(y)}{G(y)}}{\partial y_i} \geq 0
\]
\[
\iff \log \frac{R(y)}{G(y)} \text{ is non-decreasing in } y_i.
\]

for all \( y_1, y_2 > 0 \).
7. If \( s_i(y) \leq r_i(y) \) and \( \frac{s_i(y)}{r_i(y)} \) is non-decreasing in \( y_i \), then \( h_i(y) \leq k_i(y) \), \( i=1,2 \).

Proof:

\[
\frac{s_i(y)}{r_i(y)} \text{ is non-decreasing implies } \\
\log \frac{s_i(y)}{r_i(y)} \text{ is non-decreasing and therefore,}
\]

\[
\frac{s_i'(y)}{s_i(y)} - \frac{r_i'(y)}{r_i(y)} \geq 0, \quad (4.9)
\]

where, primes denote the derivative with respect to \( y_i \). Since \( s_i(y) \leq r_i(y) \) and the inequality (4.9) leads to

\[
\frac{1+s_i'(y)}{s_i(y)} - \frac{1+r_i'(y)}{r_i(y)} \geq 0
\]

or

\[ k_i(y) \geq h_i(y) \]

8. \( X \) is IFR if and only if \( Y \) is NBU(VS)

Proof:

By definition (Buchanan and Singpurwalla (1977))
Y is NBU (VS) \iff G(y_1 + t_1, y_2 + t_2) \leq G(y_1, y_2) G(t_1, t_2) for all \( y_1, y_2, t_1, t_2 > 0 \)

\[
\frac{R(x_1 + y_1 + t_1, x_2 + y_2 + t_2)}{R(x_1, x_2)} \leq \frac{R(x_1 + y_1, x_2 + y_2)}{R(x_1, x_2)} \leq \frac{R(x_1 + t_1, x_2 + t_2)}{R(x_1, x_2)}
\]

Thus, \( Y \) is NBUFR.

Since the above inequality is true for all \( x_i, y_i \) and \( t_i > 0 \), we have

\[
\frac{R(x_1 + t_1, x_2 + t_2)}{R(x_1, x_2)} \text{ is decreasing in } x_1, x_2.
\]

Therefore, from Marshall (1975), \( X \) is IFR.

The converse part follows by retracing the above steps.

9. \( X \) is IFR if and only if \( Y \) is NBUFR.

Proof:

A bivariate random vector \( X \) is said to have bivariate new better than used in failure rate (NBUFR) property if and only if
\[ h_1(x_1, o) \geq h_1(o, o) \]
and
\[ h_2(o, x_2) \geq h_2(o, o) \quad \text{for all } x_1, x_2 > 0. \]

Y is NBUFR \[ \iff \quad k_1(y_1, o) \geq k_1(o, o), \quad \text{and} \]
\[ k_2(o, y_2) \geq k_2(o, o) \]
\[ \iff \quad h_1(x_1+y_1, x_2) \geq h_1(x_1, x_2), \quad \text{and} \]
\[ h_2(x_1, x_2+y_2) \geq h_2(x_1, x_2). \]

Thus, X is IFR (Johnson and Kotz (1975)).

10. X is DMRL(1) and \( \frac{s_i(y)}{R_i(y)} \) is non-decreasing in \( y_i \),
i=1,2 together implies X is IFR.

Proof:

X is DMRL(1) \( \iff \quad \frac{s_i(x)}{R_i(x)} \) from (4.7)

The inequality (4.7) and \( \frac{s_i(y)}{R_i(y)} \) is non-decreasing in \( y_i \) together leads,
\[ h_i(y) \leq k_i(y) \quad \text{from property 7}. \]

Thus, X is IFR, from (4.6).
4.4 Characterizations by properties of RLD.

In this section, we establish certain transformations under which the RLD's of some bivariate distributions are identical with the original distributions. The result in this section is to appear in Sankaran and Nair (1992 a).

Theorem 4.1.

The necessary and sufficient condition that $G(y;x)$ satisfies the relationship

$$G(u(x)y; x) = R(y)$$

for all $x, y > 0$, where

$$u(x) = \frac{r_1(x)}{\tilde{r}_1(o)} = \frac{r_2(x)}{\tilde{r}_2(o)}$$

is that $X$ is distributed either as $P(a_1, a_2, o, c)$ or $F(p_1, p_2, o, d)$ or $E(a_1, a_2, o)$.

Proof:

Assume that condition (4.10) is satisfied. Then from (4.1) we can write,

$$R(x_1 + y_1 u(x), x_2 + y_2 u(x)) = R(y_1, y_2)R(x_1, x_2).$$
Denoting $x_1 + y_1 u(x) = s$, $x_2 + y_2 u(x) = t$, we have

on differentiating (4.12),

$$\frac{\partial R}{\partial s} \left[1 + y_1 \frac{\partial}{\partial x_1} \right] + \frac{\partial R}{\partial t} y_2 \frac{\partial u}{\partial x_1} = R(y_1, y_2) \frac{\partial R(x_1, x_2)}{\partial x_1}, \quad (4.13)$$

$$\frac{\partial R}{\partial s} u(x) = \frac{\partial R(y_1, y_2)}{\partial y_1} R(x_1, x_2), \quad (4.14)$$

$$\frac{\partial R}{\partial s} y_1 \frac{\partial u}{\partial x_2} + \frac{\partial R}{\partial t} \left[1 + y_2 \frac{\partial}{\partial x_2} \right] = R(y_1, y_2) \frac{\partial R(x_1, x_2)}{\partial x_2}; \quad (4.15)$$

and

$$\frac{\partial R}{\partial t} u(x) = \frac{\partial R(y_1, y_2)}{\partial y_2} R(x_1, x_2) \quad (4.16)$$

for all $y_1, y_2 > 0$.

Because of the absolute continuity of $R$, the partial derivatives mentioned above exist.

When $y_2$ tends to zero in (4.13),

$$\frac{\partial R}{\partial s} \left[1 + y_1 \frac{\partial}{\partial x_1} \right] = R(y_1, 0) \frac{\partial R(x_1, x_2)}{\partial x_1}. \quad (4.17)$$

Eliminating $\frac{\partial R}{\partial s}$ between (4.14) and (4.17), we have

$$[u(x)]^{-1} (1 + y_1 \frac{\partial u}{\partial x_1}) = R(y_1, 0) \frac{\partial R(x_1, x_2)}{\partial x_1} / [R(x_1, x_2) \frac{\partial R}{\partial y_1}].$$
and whence, by making \( y_1 \) tend to zero,

\[
[u(x)]^{-1} = \frac{\partial R}{\partial x_1} / [R(x_1,x_2)(\frac{\partial R}{\partial y_1}) y_1=0].
\]

The last two equations lead to

\[
1 + y_1 \frac{\partial u}{\partial x_1} = R(y_1,0)(\frac{\partial R}{\partial y_1})^{-1} [(\frac{\partial R}{\partial y_1}) y_1=0]. \quad (4.18)
\]

The right side of (4.18) is independent of both \( x_1 \) and \( x_2 \) and therefore,

\[
\frac{\partial u}{\partial x_1} = a_1, \text{ a constant.}
\]

Similarly,

\[
\frac{\partial u}{\partial x_2} = a_2.
\]

The above two conditions hold good if and only if for some \( k \),

\[
u(x) = a_1 x_1 + a_2 x_2 + k.
\]

Since \( u(0,0) = 1 \), \( k=1 \).

Thus,

\[
r_i(x_1,x_2) = (1+a_1 x_1+a_2 x_2) r_i(0), i=1,2.
\]
The three admissible cases about the values of $a_1$ and $a_2$ are that either $a_1 = a_2 = 0$, or $a_1 > 0$, $a_2 > 0$ or $a_1 < 0$, $a_2 < 0$. In the first case $r(x_1, x_2) = (r_1, r_2)$ where both $r_1$ and $r_2$ are independent of $X_1$ and $X_2$. This is true if and only if $X$ is distributed as the product of two independent exponentials (Nair and Nair, 1988). When both $a_1$ and $a_2$ are positive reals, we find from the equation (1.28) that

$$R(x_1, x_2) = (1 + a_1 x_1)^{a_2 r_2 - a_1 r_1} (1 + a_1 x_1 + a_2 x_2)^{-r_2 a_2}.$$  

For $R(x_1, x_2)$ given above to be a proper survival function, one should have $a_2 r_2 < a_1 r_1$. Now, the roles of $r_1(x_1, x_2)$ and $r_2(x_1, x_2)$ in (4.19) can be interchanged and this leads to the condition $a_1 r_1 < a_2 r_2$ and hence $a_1 r_1 = a_2 r_2 = c$, with $c > 0$. Thus $X$ follows $P(a_1, a_2, 0, c)$. When $a_1, a_2 < 0$, taking $a_1 = -p_1$ and $a_2 = -p_2$, $p_1, p_2 > 0$ and repeating the above steps,

$$R(x_1, x_2) = (1 - p_1 x_1 - p_2 x_2)^d,$$  

so that $X$ is $F(p_1, p_2, 0, d)$. The converse part follow from examples 1, 3 and 4.
As a consequence of Theorem 4.1, we get the following characterization theorem connected with the proportionality of the two components of vector valued failure rate \( h(x_1,x_2) \).

Theorem 4.2.

The relationship,

\[
a_2 h_1(x_1,x_2) = a_1 h(x_1,x_2)
\]

(4.20)

is satisfied for all \( x_1,x_2 > 0 \) if and only if \( X \) follow one of the distributions in Theorem 4.1.

Proof:

Eliminating \( \frac{\partial R}{\partial s} \) and \( \frac{\partial R}{\partial t} \) from equations (4.13) through (4.16) we have for \( i,j=1,2, i\neq j \)

\[
\frac{1}{R(y_1,y_2)} \frac{\partial R}{\partial y_j} \frac{R(x_1,x_2)}{u(x_1,x_2)} \left[ 1 + \frac{\partial u}{\partial x_1} y_1 + \frac{\partial u}{\partial x_2} y_2 \right]
\]

\[
= \frac{\partial R}{\partial x_j} [1 + \frac{\partial u}{\partial x_i} y_1] - \frac{\partial R}{\partial x_i} \frac{\partial u}{\partial x_j} y_i
\]

and whence

\[
\left( \frac{\partial R}{\partial x_2} \frac{\partial u}{\partial x_1} - \frac{\partial R}{\partial x_1} \frac{\partial u}{\partial x_2} \right) \left( \frac{\partial R}{\partial y_2} y_2 + \frac{\partial R}{\partial y_1} y_1 \right) = 0.
\]
Since the second expression in the above product non-zero, we find

\[ \frac{\partial R}{\partial x_1} \frac{\partial u}{\partial x_2} = \frac{\partial R}{\partial x_2} \frac{\partial u}{\partial x_1} \]

or

\[ a_2 h_1(x_1, x_2) = a_1 h_2(x_1, x_2) \]

Since the equations (4.13), (4.14), (4.15) and (4.16) taken together characterize the three distributions given in Theorem 4.1, the relationship (4.20) is satisfied for those three distributions.

Conversely, when \( X \) follows \( E(a_1, a_2, o) \)

\( (P(a_1, a_2, o, c); F(p_1, p_2, p, d)) \), the ratio \( \frac{h_1(x_1, x_2)}{h_2(x_1, x_2)} \) is

\[ 1\left( \frac{a_1}{a_2}; \frac{p_1}{p_2} \right) \]. This completes the proof.

In the following theorem we address to a more general question, by appealing to the functions

\[ u_1(x) = \frac{r_1(x)}{r_1(o)} \text{ and } u_2(x) = \frac{r_2(x)}{r_2(o)} \]

to provide some characterizations.

**Theorem 4.3.**

The RLD admits the conditions
\[ G(y_1 u_1(x);x) = R_1(y_1) \quad i=1,2, \quad (4.21) \]

where, \( R_1(y_1) \) is the marginal survival function of \( y_1 \), if and only if \( X \) follows one of the distributions specified in equations (3.2), (3.3) and (3.4).

**Proof:**

When \( X \) follows exponential, \( E(\alpha_1, \alpha_2, \Theta) \), for \( i=1 \)

\[ u_1(x) = \alpha_1 (\alpha_1 + \Theta x_2)^{-1} \]

and

\[ G_1(y_1; x) = e^{-(\alpha_1 + \Theta x_2)x_1}, \]

so that

\[ G_1(y_1 u_1(x); x) = e^{-\alpha_1 y_1} \]

\[ = R_1(y_1), \]

Where as for Pareto \( P(a_1, a_2, b, c) \)

\[ u_1(x) = \frac{(1+a_1 x_1 + a_2 x_2 + bx_1 x_2)a_1}{a_1 + bx_2} \]

and

\[ G_1(y_1; x) = \left[ 1+(a_1+bx_2)y_1/(1+a_1 x_1+a_2 x_2+bx_1 x_2) \right]^{-c} \]

giving

\[ G_1(y_1 u_1(x); x) = (1+a_1 y_1)^{-c} \]

\[ = R_1(y_1). \]
In the finite range case $F(p_1, p_2, q, d)$

$$u_1(x) = \frac{p_1 - q x_2}{1 - p_1 x_1 - p_2 x_2 + q x_1 x_2}$$

and

$$G_1(y_1; x) = \left[1 - (p_1 - q x_2) y_1/(1 - p_1 x_1 - p_2 x_2 + q x_1 x_2)\right]^d.$$ 

Thus,

$$G_1(y_1 u_1(x); x) = (1 - p_1 y_1)^d = R_1(y_1).$$

To prove the only if part, we proceed along the lines of Theorem 4.1. The equation (4.21) is equivalent, for $i=1$, to

$$R(x_1 + y_1 u_1(x), x_2) = R(x_1, x_2) R_1(y_1). \quad (4.22)$$

From (4.17) and (4.14)

$$\frac{\partial R}{\partial s} (1 + y_1 \frac{\partial u_1}{\partial x_1}) = \frac{\partial R}{\partial x_1} R_1(y_1) \quad (4.23)$$

or

$$\frac{\partial R}{\partial s} u_1(x_1, x_2) = R(x_1, x_2) \frac{\partial R}{\partial y_1}. \quad (4.24)$$

Dividing the equation (4.23) by (4.24), we get
\[ \frac{\partial u_1}{\partial x_1} \frac{1+Y_1}{1+Y_1} = R_1(Y_1) \frac{\partial R}{\partial y_1} \frac{\partial R}{\partial y_1} y_1 = 0 \]

which implies

\[ \frac{\partial u_1}{\partial x_1} = a_1 \]

or

\[ u_1(x) = a_1 x_1 + B_1(x_2). \]

Similarly,

\[ u_2(x) = a_2 x_2 + B_2(x_1). \]

Using the same arguments in Theorem 4.1, we have

\[ a_1 r_1(o) = a_2 r_2(o) = l. \]

Thus the MRL function is of the form

\[ r(x) = (l x_1 + m_1(x_2), l x_2 + m_2(x_1)) \]

which is a characteristic property of the family of distributions mentioned in Theorem 4.3 and the proof is complete.

Corollaries:

1. Taking \( i=1 \) and allowing \( x_2 \) to tend to zero, the conditions stated in Oakes and Dasu (1990) that
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characterizes the univariate exponential, Pareto and finite range distributions result.

2. \( u_1(x) = u_2(x) \) in Theorem 4.3 characterizes the reduced models of Theorem 4.1.