Chapter 3

BIVARIATE MODELS WITH LINEAR MEAN RESIDUAL LIFE COMPONENTS

3.1. Introduction

The discussions in the previous chapter were confined to univariate life distributions and representations that enable their unique determination. In addition to these, there exist a large class of problems in reliability that necessitate the generation and use of multivariate distributions like the study of multicomponent systems where a random variable has to be associated with the lifetime of each component. The system works in unison with the various components and therefore the life distributions of all of them, individually and collectively would be of interest in evaluating the performance of the system. Bivariate distributions are often singled out for detailed study in view of its special relevance to systems consisting of two components. The pre-occupation with multivariate normal distribution by most researchers of early days coupled with the widespread belief that normal law holds either exactly or approximately in many natural phenomena led to a slow progress in generating non-normal multivariate models. Consistent with this
general trend, the literature on life distributions in higher dimensions is limited and even those models that have already surfaced needs detailed scrutiny.

Galambos and Kotz (1978) lists the different methods of constructing a multivariate distribution as

(a) extending a univariate system to the multivariate set up,

(b) deriving the model through the mathematical relations between the joint distribution and its marginals,

(c) postulating a multivariate form via extending the functional form of the corresponding univariate family, and

(d) extending a meaningful characterizing property of the univariate case to the multivariate case and then deriving the distribution characterized by such a property.

As already mentioned a desirable option in many modelling problem is to look out for the physical properties of the system and to extract a probability distribution consistent with them. The failure rate, MRL function etc. being the summary characteristics of the failure patterns, in the present chapter we
postulate the form of such functions in developing the required models. A functional form that is simple in nature and at the same time can depict the different patterns of ageing via decreasing, increasing or constant mean residual life will serve our purpose. Motivated by the result of Kotz and Shanbhag (1980) in the univariate case that a linear MRL function or a reciprocal linear failure rate function characterizes the family consisting of the exponential, Pareto and finite range distributions, we concentrate on a generalization of this form to generate a class of corresponding bivariate distributions and examine its properties and applications in reliability analysis. The univariate exponential, Pareto and finite range models have been elaborately explored in literature in the context of characterization, reliability modelling and a variety of other applications. Extensions of these models by generalising different characteristic properties in the univariate case, may often lead to different bivariate versions. In other words the bivariate models, we are seeking, need not inherit extended versions of all the univariate characteristics. Thus it is our endeavour here to examine the nature of the characterizations brought out by the new models.
In the rest of this study an attempt is made to resolve some of these problems.

3.2. The Models

Let $X = (X_1, X_2)$ be a bivariate random vector admitting absolutely continuous distribution function with respect to Lebesgue measure in the positive octant $H_2^+ = \{(x, y) \mid x, y > 0\}$ of the two dimensional Euclidean space $R_2$. For $(x_1, x_2) \in R_2^+$, the MRL $r(x_1, x_2)$ defined in equation (1.24) is vector valued with components

$$r_i(x_1, x_2) = E[X_i - x_i | X_1 > x_1, X_2 > x_2].$$

In the following theorem we identify the models that are uniquely determined by the fact that $r_i(x_1, x_2)$ is linear in $x_i$.

Theorem 3.1.

The random vector $X$ defined above has an MRL function $r(x_1, x_2)$ with components of the form

$$r_i(x_1, x_2) = Ax_i + B_i(x_j) \quad i, j = 1, 2, i \neq j,$$  \hspace{1cm} (3.1)

where, $B_i(x_j) > 0$ for all $x_j > 0$ if and only if it is distributed as
(a) Gumbel's (1960) bivariate exponential distribution with survival function,
\[ R(x_1, x_2) = e^{-\alpha_1 x_1 - \alpha_2 x_2 - \theta x_1 x_2} \]
where \( \alpha_1, \alpha_2 > 0, x_1, x_2 > 0 \), \( 0 < \theta \leq \alpha_1 \alpha_2 \) when \( A = 0 \).

(b) Bivariate Pareto distribution specified by
\[ R(x_1, x_2) = \frac{c}{(1 + a_1 x_1 + a_2 x_2 + b x_1 x_2)^c} ; x_1, x_2 > 0 \]
where \( a_1, a_2, c > 0 \), \( 0 < b \leq (c+1)a_1 a_2 \) when \( A > 0 \).

(c) Bivariate finite range model with
\[ R(x_1, x_2) = (1 - p_1 x_1 - p_2 x_2 + q x_1 x_2)^d \]
where \( c = \frac{1}{A} > 0 \). On equating the expressions in (3.5) and (3.6),
\[ \frac{x_1}{p_1} + \frac{x_2}{p_2} = \frac{x_1}{q} + \frac{x_2}{1 - q} \]
where \( \mu_1 = B_1(0+), \mu_2 = B_2(x_1) \), \( \gamma = 1, 2 \).
Proof:

The sufficiency part will be first established.

When $A = 0$, (3.1) becomes,

$$r_i(x_1, x_2) = B_i(x_j), \ i, j = 1, 2, \ i \neq j.$$  

That this form characterizes the Gumbel's bivariate exponential distribution is proved in Nair and Nair (1988). We now turn to the other two cases. Suppose that $A > 0$. Using the formulas in equations (1.28) and (1.29), connecting the survival function and the components of the MRL,

$$R(x_1, x_2) = (1 + \frac{A x_1}{B_1(0)})^{-c} (1 + \frac{A x_2}{B_2(x_1)})^{-c}$$  

(3.5)

and also,

$$R(x_1, x_2) = (1 + \frac{A x_2}{B_2(0)})^{-c} (1 + \frac{A x_1}{B_1(x_2)})^{-c}$$  

(3.6)

where, $c = \frac{1}{A} > 0$.

On equating the expressions in (3.5) and (3.6),

$$\frac{x_1}{\mu_1} + \frac{x_2}{B_2(x_1)} + \frac{A x_1 x_2}{\mu_1 B_2(x_1)} = \frac{x_2}{\mu_2} + \frac{x_1}{B_1(x_2)} + \frac{A x_1 x_2}{\mu_2 B_1(x_2)}$$  

(3.7)

where, $\mu_i = B_i(0^+), \ i = 1, 2.$
Dividing the equation (3.7) by $x_1 x_2$ and simplifying yield,

$$\frac{1}{x_1} + \frac{1}{x_2 B_2(x_1)} + \frac{A}{\mu_1 B_2(x_1)} = \frac{-1}{x_2 \mu_1} + \frac{1}{x_2 B_1(x_2)} + \frac{A}{\mu_2 B_1(x_2)}.$$  \hspace{1cm} (3.8)

That the last equation holds for all $x_1, x_2 > 0$ would, however, mean that

$$\frac{A}{\mu_1 B_j(x_1)} + \frac{1}{x_1 B_j(x_1)} - \frac{1}{\mu_1 x_1} = 0,$$ \hspace{1cm} (3.9)

Hence for $i=1$,

$$B_2(x_1) = \frac{(\mu_1 + A x_1) \mu_2}{\mu_1 (1 + \Theta x_1 \mu_2)}.$$ \hspace{1cm} (3.10)

Substituting (3.9) in (3.5),

$$R(x_1, x_2) = (1 - A x_1^{-c} x_2^{-c} (1 + A x_1)^{-c} (1 + A x_2)^{-c})$$ \hspace{1cm} (3.11)

and

$$R(x_1, x_2) = (1 + a_1 x_1 + a_2 x_2 + b x_1 x_2)^{-c}$$ \hspace{1cm} (3.12)

with $a_i = \frac{A}{\mu_i} i = 1, 2$, and $b = A \Theta$. 

This gives $a_1, a_2, c > 0$. From the condition $R(x_1, x_2) = R(x_1, x_2)$ for all $x_1, x_2$, we get $c > 0$. The probability density function corresponding to (3.3) is
Now we derive the conditions on the parameters that render (3.3) the status of a survival function. When (3.3) is a joint survival function, \( R(x_1, o) \) and \( R(o, x_2) \) are the marginal survival functions of \( X_1 \) and \( X_2 \).

This gives \( a_1, a_2, c > 0 \). From the condition \( R(x_1, o) > R(x_1, x_2) \) for all \( x_1, x_2 \), we get \( b > 0 \). The probability density function corresponding to (3.3) is

$$f(x_1, x_2) = c(c(a_1 + bx_2)(a_2 + bx_1) + a_1 a_2 - b)(1 + a_1 x_1 + a_2 x_2 + bx_1 x_2)$$

(3.10)

Since \( f(o, o) > 0 \), \( b < (c + 1)a_1 a_2 \).

This completes the proof of the sufficiency part when \( A > 0 \).

When \( A < 0 \),

$$R(x_1, x_2) = (1 - \frac{A x_1}{B_1(o)})^d (1 - \frac{A x_2}{B_2(x_1)})^d$$

(3.11)

and

$$R(x_1, x_2) = (1 - \frac{A x_2}{B_2(o)})^d (1 - \frac{A x_1}{B_1(x_2)})$$

(3.12)

where,

$$d = \frac{-1}{A} > 0.$$
Identifying (3.11) and (3.12) and then simplifying the resulting equation as in the previous case renders

\[
\frac{1}{x_i \mu_j} - \frac{1}{x_i B_j(x_i)} + \frac{A}{\mu_i B_j(x_i)} = \beta; \quad i, j = 1, 2. \quad i \neq j
\]

and

\[
B_1(x_2) = \frac{(\mu_2 - A x_2) \mu_1}{\mu_2 (1 - \mu_1 \beta x_2)}.
\]

Substituting (3.13) in (3.12), the survival function of \( X \) is

\[
R(x_1, x_2) = (1 - p_1 x_1 - p_2 x_2 + q x_1 x_2)^d
\]

where

\[
p_i = \frac{A}{\mu_i} \quad i = 1, 2.
\]

and

\[
q = A \beta.
\]

The marginal density of \( X_i \) is

\[
f_i(x_i) = dp_i(1 - p_i x_i)^{d-1}
\]

since \( f_i(\varnothing) > 0, \quad p_i > 0 \quad i = 1, 2. \)

The probability density function corresponding to (3.4) is given by
\[ f(x_1, x_2) = d[q-p_1 p_2 + d(q x_1 - p_2)(q x_2 - p_1)] \]
\[ (1 - p_1 x_1 - p_2 x_2 + q x_1 x_2)^{d-2}. \]  

(3.14)

The condition \( f(o, o) \geq 0 \) implies

\[ 1 - d \leq \frac{q}{p_1 p_2}. \]

(3.10)

Since \( R(x_1, x_2) \leq R(x_1, o) \) for all \( x_1, x_2 > 0 \), one must have \( \frac{q}{p_1 p_2} \leq 1 \). Conversely, the MRL functions of the Gumbel's bivariate exponential, bivariate Pareto and the bivariate finite range distributions are respectively

\[ \gamma_E(x_1, x_2) = \left( \frac{1}{\alpha_1 + \theta_2 x_2}, \frac{1}{\alpha_2 + \theta x_1} \right), \]  

(3.15)

\[ \gamma_P(x_1, x_2) = \left( \frac{x_1}{c-1} + \frac{1+\alpha x_2}{(\alpha_1 + bx_2)(c-1)}, \frac{x_2}{c-1} + \frac{1+\alpha x_1}{(\alpha_2 + bx_1)(c-1)} \right), \]  

(3.16)

and

\[ \gamma_F(x_1, x_2) = \left( -\frac{x_1}{d+1} + \frac{1-p_1 x_2}{(p_1-q x_2)(d+1)}, \frac{-x_2}{d+1} \right), \]

(3.17)
It is easy to see that the conditions of the theorem are necessary.

Corollaries.

(i) When $\Theta = 0$ in (3.2),

\[ R(x_1, x_2) = e^{-\alpha_1 x_1 - \alpha_2 x_2}, \quad (3.18) \]

showing that $X_1$ and $X_2$ are independent and exponentially distributed. In this case,

\[ r(x_1, x_2) = (\alpha_1^{-1}, \alpha_2^{-1}), \]

a constant vector, characterizes (3.18).

(ii) Setting $b = 0$, in (3.3)

\[ R(x_1, x_2) = (1 + a_1 x_1 + a_2 x_2)^{-c}, \quad x_1, x_2 > 0, \quad c > 0 \quad (3.19) \]

which is the model obtained in Lindley and Singpurwalla (1986) under a different set of conditions. The MRL function, characteristic of the model, has form

\[ r(x_1, x_2) = \left( \frac{x_1}{c-1} + \frac{1+a_2 x_2}{a_1}, \quad \frac{x_2}{c-1} + \frac{1+a_1 x_1}{a_2} \right), \]

so that the components are linear in both $x_1$ and $x_2$ with positive coefficients.
(iii) For $b = a_1 a_2$ in (3.3), $X_1$ and $X_2$ have independent Pareto II distributions with survival functions

$$R_i(x_i) = (1+a_i x_i)^{-c}, \ i=1,2,$$

and $i$th component of the MRL is linear in $x_i$ alone.

(iv) Taking $q = 0$ in (3.4),

$$R(x_1,x_2) = (1-p_1 x_1-p_2 x_2)^d.$$

We have the bivariate finite range model given in Roy (1989). The components of the MRL function are again linear with negative coefficients.

(v) In the case $q=p_1 p_2$, the survival function of $X=(X_1,X_2)$ is the product of survival functions of univariate finite range distributions and thus $X_1$ and $X_2$ are independent.

(vi) The bivariate failure rate defined in equation (1.17) for these models of theorem is of the form

$$h(x_1,x_2) = \left( (Cx_1+D_1(x_2))^{-1}, (Cx_2+D_2(x_1))^{-1} \right), \ (3.20)$$
Since,
\[ h_1(x_1, x_2) = \frac{1}{1 + \frac{\partial r_1}{\partial x_1}} \cdot \frac{\partial r_1}{\partial x_1}, \]
with
\[ C = A(1+A)^{-1} \]
and
\[ D_1(x_j) = B_1(x_j)(1+A)^{-1}. \]

The forms of the failure rates in the different cases considered in corollaries i to v follow immediately from the transformations in (3.21).

Note:

The bivariate exponential distribution with \( \alpha_1 = \alpha_2 = 1 \) and its properties are discussed in Gumbel (1960). The bivariate Pareto has appeared in a different context in Hutchinson (1979) and is a particular case of the bivariate Burr distribution cited in Johnson and Kotz (1972).

3.3. Bivariate Pareto Distribution

Inspite of the appearance of the form of the density function of the bivariate Pareto distribution described...
in section 3.2 in some earlier investigations as cited at the end of the last section, the distributional properties do not seem to have been discussed anywhere. Accordingly we explore this aspect in the rest of this section and list the salient properties.

(1) The marginal distribution of $X_i$ is

$$f_i(x_i) = c a_i (1 + a_i x_i)^{-(c+1)}, \quad i = 1, 2,$$

which is of univariate Pareto II form whose properties are discussed in Arnold (1983). In particular the mean and variance of $X_i$ are

$$E(X_i) = \frac{1}{a_i (c-1)} \quad (3.22)$$

and

$$V(X_i) = \frac{c}{(c-1)^2 (c-2) a_i^2}.$$

(2) The conditional distribution of $X_i$ given $X_j = x_j$ $i, j = 1, 2, i \neq j$ as

$$g_i(x_i|x_j) = \frac{[c(a_1 + bx_2)(a_2 + bx_1) + a_1 a_2 - b]}{a_j (1 + a_j x_j)^{c-1}} (1 + a_1 x_1 + a_2 x_2 + bx_1 x_2)^{-c-2}.$$

Once this form of the conditional probability density
function is assumed, it can be shown that $X_1$ is Pareto type II if and only if $X_2$ is of the same form (Hitha and Nair, 1991).

(3) By direct calculation from the conditional density given above,

$$E[X_1|X_j=x_j] = \frac{a_j c (c-1)(a_i+bx_j)^2(1+a_j x_j)}{[a_j c (a_i+bx_j)+b-a_1 a_2]}.$$  

(3.23)

Accordingly, for $0 < b < (c+1)a_1 a_2$, the regression curves are non-linear. Specialising for $b=0$, equation (3.23) becomes,

$$E[X_1|X_j=x_j] = \frac{1+a_j x_j}{(c-1)a_1},$$

and the regression turns out to be linear. The two lines intersect at

$$(- (a_1+a_2)^{-1}, c^{-1}(a_1+a_2)^{-1}),$$

which is not $(E(X_1), E(X_2))$ as in the bivariate normal case. Further, the coefficient of correlation in this situation is $c^{-1}$ which is always positive.
Also,  

\[ E(X_1X_2) = \int_0^\infty \int_0^\infty x_1x_2 \, c[c(a_1+bx_2)(a_2+bx_1)+a_1a_2-b] \]

\[ (1+a_1x_1+a_2x_2+bx_1x_2)^{-c-2} \, dx_1 \, dx_2, \]

\[ = \frac{b-a_1a_2}{c-1} \int_0^\infty \frac{x_1}{(a_2+bx_1)^2(1+a_1x_1)c} \, dx_1 \]

\[ + \frac{a_1c}{(c-1)} \int_0^\infty \frac{x_1}{(a_2+bx_1)(1+a_1x_1)c} \, dx_1. \]

From Erdelyi et al. (1954)

\[ \int_0^\infty \frac{x^{v-1}}{(a+x)^{\mu}(x+y)} \, dx = \frac{\Gamma_v \Gamma_{\mu+\nu} \Gamma_{\nu-\rho} \Gamma_{\mu+\rho}}{\Gamma_{\mu+\nu-\rho} a^\mu} F(\mu, \nu; \mu+\rho; \frac{1-y}{a}) \quad (3.24) \]

where, \( F(p,q,r;t) \) represent the hypergeometric function,

\[ F(p,q,r;t) = \sum_{n=0}^\infty \frac{p+n}{p} \frac{q+n}{q} \frac{t^n}{n!} \]

With the aid of (3.24),

\[ \int_0^\infty \frac{x_1 \, dx_1}{(a_2+bx_1)^2(1+a_1x_1)c} = \frac{F(2,2,2+c,y)}{a_1^2 a_2^2 c(c+1)} \]

(3.26)
\[ \int_0^\infty \frac{x_1^{c-1}dx_1}{(a_2+bx_1)(1+a_1x_1)^c} = \frac{F(1,2,c+1,y)}{a_1^2 a_2 c(c-1)} \]

Thus,
\[ E(X_1X_2) = \frac{b-a_1a_2}{a_1^2 a_2^2 c(c+1)(c-1)} F(2,2,2+c,y) + \frac{a_1c}{(c-1)^2 ca_1^2 a_2} F(1,2,c+1,y) \tag{3.25} \]

From (3.22) and (3.25), the coefficient of correlation between \( X_1 \) and \( X_2 \) for the general model turns out to be
\[
= (c-2)c^{-1}[F(1,2,c+1,y)-1-y(c-1)(c+1)^{-1}c^{-1}F(2,2,c+2y)],
\]
where, \( y = \frac{a_1a_2-b}{a_1^2 a_2} \).

(4) From the reliability point of view, the conditional distribution of \( X_i \) given \( X_j = x_j \) is not of particular significance. It is of more interest to study the conditional distributions of \( X_i \) given \( X_j > x_j \), which has the density function,
\[ f_i(X_i|X_j > x_j) = ca_i(x_j)[1+a_i(x_j)x_i]^{-c-1} \]
with, \( a_i(x_j) = \frac{a_i + bx_j}{1 + a_jx_j} \).

Equation (3.26) reveals that such conditional distributions again are of Pareto II form. This fact enables the characterization of the bivariate Pareto distribution (Hitha and Nair (1991)).

(5) From \( h_i(x_1, x_2) = \frac{c(a_i + bx_i)}{1 + a_1x_1 + a_2x_2 + bx_1x_2} \), it is evident that \( h_i(x_1, x_2) \) decreases on \( x_i \) and accordingly the distribution of \( X = (X_1, X_2) \) is bivariate decreasing failure rate in the sense of Johnson and Kotz (1975).

Similarly, using the MRL components,

\[
\rho_i(x_1, x_2) = (c-1)^{-1} \left( x_i + \frac{1 + a_i x_j}{a_i + b x_j} \right),
\]

one can see that \( \rho_i(x_1, x_2) \) is an increasing function of \( x_i \). Therefore \( X \) is IMRL (2) (Zahedi (1985)).

(6) The Basu's failure rate defined in equation (1.13)

\[
a(x_1, x_2) = \frac{c[c(a_1 + bx_2)(a_2 + bx_1) + a_1a_2 - b]}{(1 + a_1x_1 + a_2x_2 + bx_1x_2)^2},
\]
(7) The distribution has modified MRL vector
\((\text{equations } (1.33) \text{ and } (1.34))\)

\[
\left( \frac{1+a_1 x_1 + a_2 x_2 + bx_1 x_2}{(c-1)(a_1 + bx_2)}, \frac{1+a_2 x_2}{(c-1)a_2} \right).
\]

(8) If \((X_1^{(i)}, X_2^{(i)}), i=1,2,...,n\) are independent
and identically distributed random variables following bivariate Pareto, then \((\min X_1^{(i)}, \min X_2^{(i)})\)
also has the same distribution.

(9) Generalised versions of several bivariate distributions that could be meaningful from the context of
reliability such as Pareto of Mardia (1962), version of
Burr model of Johnson and Kotz (1972), logistic of
Satterthwaite and Hutchinson (1978) are obtainable
through monotonic transformations, described in
Nayak (1987), on \((3,3)\). For example, setting \(Y_i = X_i + a_i^{-1}\)
will result in the generalised Mardia family specified
by
\[
R(y_1, y_2) = (1 + \frac{\theta}{a_1 a_2})^c [a_1 y_1 + a_2 y_2 + \theta y_1 y_2 - 1]^{-c}, y_1 \geq a_1^{-1}
\]
after some obvious reparametrisations.
3.4. **Physical Interpretation of the Bivariate Pareto Model**

In the study of reliability of series and parallel systems assessment of their component reliabilities is an important practical problem. The component reliabilities assessed at the manufacturing stage or at the quality performance test stage are usually taken as the reliability expected of the components at all subsequent stages of its utility. In general, the environmental conditions under which a component is operating need not be the same as that in the laboratory where preliminary tests are performed to measure the reliability. Often a component may perform better or worse in an environment different from that of the test site. This brings in the problem of studying the effect of change in operating conditions in evaluating the reliability of a system.

On the ground that most studies on reliability do not take into consideration, the influence of the operating environment to the system, Lindley and Singpurwalla (1986), proposed a method of modelling the life lengths of the components of a system and derived a bivariate Pareto distribution that could accommodate such changes in the operating environment. The major
assumptions used in developing the model were:

(i) the components life lengths are independent exponential variables, and

(ii) the influence of the operating conditions change the original failure rates $\alpha_1$ and $\alpha_2$ to $\beta\alpha_1$ and $\beta\alpha_2$, where the uncertainty in $\beta$ is described by a gamma distribution.

The distribution obtained under (i) and (ii) is given in equation (3.19). In this formulation $\beta>1$, $\beta<1$ and $\beta=1$ respectively suggest a harsh, mild and same conditions of use as compared to the laboratory environments. The assumption of independence of the components made by Lindley and Singpurwalla (1986) is not always a reality as the life lengths of a system can depend on one another. Instead of assuming independent exponential laws for the component lives, Bandyopadhyay and Basu (1990) proposed a dependence structure among them by permitting the system to operate in a test environment consisting of shocks that lead to the Marshall-Olkin (1967) bivariate exponential distribution. In this way, they obtained a bivariate
Pareto law with survival function,

\[ R(x_1, x_2) = (1 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_{12} \max(x_1, x_2))^{-b}, \quad (3.2') \]

\[ x_1, x_2 > 0, \quad \alpha_1, \alpha_2, b > 0, \quad \alpha_{12} \geq 0, \]

which included the Lindley-Singpurwalla model as a special case when \( \alpha_{12} = 0 \).

Apart from making provision for dependent life lengths, the generalisation of Bandyopadhyay and Basu (1990) incorporates into the system the remarkable property of bivariate lack of memory defined as

\[ P[X_1 > x_1 + t, X_2 > x_2 + t | X_1 > t, X_2 > t] = P[X_1 > x_1, X_2 > x_2] \]

inherent in the Marshall-Olkin distribution. However, in situations where the simultaneous failure of components in a two-component parallel system is not to be expected, the Marshall-Olkin model is not appropriate and an absolutely continuous bivariate exponential distribution is more realistic. Accordingly, we assume that the component life lengths \( X_1 \) and \( X_2 \) follow the Gumbel's bivariate exponential law with survival function (3.2), whose dependence structure is different from
that of (3.27). In the process, the bivariate lack of memory property is sacrificed and in its place the bivariate local lack of memory defined as

\[ P[X_i > t_i + s_i | X_i > s_i, X_j > t_j] = P[X_i > t_i | X_j > t_j], \]

where \( i, j = 1, 2, i \neq j \),

which is characteristic of distribution (3.2) (see Nair and Nair (1991)) is retained. Recalling that, when \( X = (X_1, X_2) \) follows Gumbel's exponential distribution (3.2), the bivariate failure rate defined in equation (1.17) is

\[ h(x_1, x_2) = (\alpha_1 + \Theta x_2, \alpha_2 + \Theta x_1). \quad (3.28) \]

The effect of the operating environment being to increase or decrease (3.28) by a quantity \( \beta \), the new model is to be characterized by the failure rate,

\[ h_1(x_1, x_2) = (\alpha_1 \beta + \Theta x_2, \beta \alpha_2 + \Theta x_1), \]

which again leads to the Gumbel's exponential with survival function

\[ R(x_1, x_2) = \exp[-\beta(\alpha_1 x_1 + \alpha_2 x_2 + \Theta x_1 x_2)]. \quad (3.29) \]

In view of the uncertainty involved in \( \beta \), we associate
with $\beta$ a gamma distribution with parameters $m$ and $c$. After averaging (3.29) using this distribution of $\beta$, the joint survival function of $X_1$ and $X_2$ is arrived at as

$$R(x_1, x_2) = (1 + a_1 x_1 + a_2 x_2 + b x_1 x_2)^{-c},$$

where,

$$a_i = \alpha_i / m \quad \text{and} \quad b = \Theta / m, \ i = 1, 2.$$

Notice that the above survival function corresponds to the bivariate Pareto distribution analysed in the previous section and that the Lindley-Singpurwalla model is its particular case. The MRL function of the Gumbel distribution has components that are locally constant and the impact of the change in environment is that they become a linear function of the life-times. The physical interpretation we have now obtained leads some further analysis leading to the comparison of the effect of test and operating conditions on the reliability of the system.

3.5. Applications to Reliability of Series Systems

Consider a two-component series system in which the life times of the components have joint distribution
and the environment effect is described as in the previous section. The system reliability at time $t$ is then given by

$$ R_b(t) = \left(1 + a_1 t + a_2 t + b t^2\right)^{-c}. \quad (3.30) $$

For such a system the failure rate is

$$ h_b(t) = \frac{c(a_1 + a_2 + 2bt)}{(1 + a_1 t + a_2 t + b t^2)}. $$

and the mean residual life is

$$ r_b(t) = \frac{1 + a_1 t + a_2 t + b t^2}{(c-1)(a_1 + a_2 + 2bt)}. $$

The corresponding expressions for Lindley-Singpurwalla models are readily obtained by setting $b = 0$. Accordingly, the relative error in the reliability function as defined in Gupta and Gupta (1990) is

$$ e_R(t) = \frac{R_b(t) - R_0(t)}{R_0(t)} = \frac{\left(1 + a_1 t + a_2 t\right)^c}{\left(1 + a_1 t + a_2 t + b t^2\right)^c} - 1 $$

which decreases from zero to -1 as a function of $t$. 
The relative error in failure rate is

\[ e_h(t) = \frac{h_b(t) - h_o(t)}{h_o(t)} = \frac{(a_1 + a_2 + 2bt)(1 + a_1t + a_2t)}{(a_1 + a_2)(1 + a_1t + a_2t + bt^2)} - 1. \]

This is an increasing function of \( t \) with minimum value zero and maximum value unity. On the other hand, the error in mean residual life decreases in \( t \) from \( \frac{1}{2} \) to zero, as seen from the expression,

\[ e_r(t) = \frac{(a_1 + a_2)(1 + a_1t + a_2t + bt^2)}{(a_1 + a_2 + 2bt)(1 + a_1t + a_2t)} - 1 \]

or from the relationship \([1 + e_h(t)][1 + e_r(t)] = 1\).

The extreme values of the relative error in all the three cases do not depend on the parameters of the model.

A comparison of model (3.30) which incorporates the effect of environment and the counterpart

\[ R(t) = \exp[-\alpha_1 t - \alpha_2 t^2 - \Theta t^2] \quad (3.31) \]

which corresponds to the laboratory environment (\( \beta = 1 \))
seems worthwhile. The failure rate of the former is

\[ h_1(t) = \frac{c(\alpha_1+\alpha_2+ 2\Theta t)}{(m+\alpha_1 t+\alpha_2 t+\Theta t^2)} \]

and that of (3.31) is

\[ h_2(t) = (\alpha_1+\alpha_2+ 2\Theta t) \cdot \]

\[ h_1(t) \geq h_2(t) \]

according as the expression

\[ \Theta t^2+(\alpha_1+\alpha_2) t + m-c \geq 0. \]

If \( t_1 \) and \( t_2 \) are the roots of the equation obtained by setting \( \Theta t^2+(\alpha_1+\alpha_2)+m-c = 0 \), it is easy to see that one root, say \( t_2 \), is always either negative or imaginary. The last case being of no interest, the sign of above equation depends on the other root

\[ t_1 = (2\Theta)^{-1}[-(\alpha_1+\alpha_2)+\sqrt{(\alpha_1+\alpha_2)^2- 4\Theta(m-c)}] \cdot \]

Since \( E(\beta) = c/m \), this shows that \( h_1(t) \geq h_2(t) \) according as \( E(\beta) \geq 1 \), giving conditions for a harsher, same or milder operating environment, irrespective of the values of the parameters \( \alpha_1, \alpha_2 \) and \( \Theta \).

These results in the last four sections are taken from Sankaran and Nair (1992 b).
3.6. **Bivariate Finite Range Distribution**

The importance of the univariate finite range distribution in the analysis of life time data is given in Mukherjee and Islam (1983) who observe that "statistical theory does not restrict a finite upper limit to the life time of an equipment and many failure time distribution are defined over the range \((0, \infty)\), but the designed life time of equipment should only be finite".

Further, since life tests are usually censored or truncated, the observed life time of equipment varies over only a finite range. It is therefore, worthwhile to look at a generalisation of the finite range distribution in higher dimensions which is precisely the model (3.4) derived in Section 3.2.

The properties of this model bears a very close resemblance to that of the bivariate Pareto distribution especially in the expressions of the various population characteristics. However, the reliability characteristics is seen to behave more or less in the opposite sense.
Properties.

1. The probability density function corresponding to (3.4) is

\[ f(x_1, x_2) = d(1-p_1 x_1 - p_2 x_2 + q x_1 x_2)^{d-2}[d(p_1-q x_2)(p_2-q x_1)
+ q-p_1 p_2] . \]

2. The marginal distributions of \( X_i \) are

\[ f_i(x_i) = dp_i(1-p_i x_i)^{d-1} d>0, 0<x_i<\frac{1}{p_i}, i=1,2 \]

which is univariate finite range model.

Further,

\[ E(X_i) = \frac{1}{p_i(d+1)} \]  

and \[ V(X_i) = \frac{d}{(d+1)^2(d+2)p_i^2} . \]

3. The conditional distribution of \( X_i \) given \( X_j=x_j \) is

\[ g_i(x_i|x_j) = \frac{(1-p_1 x_1 - p_2 x_2 + q x_1 x_2)^{d-2}[d(p_1-q x_2)(p_2-q x_1)+q-p_1 p_2]
}{p_j(1-p_j x_j)^{d-1}}, \]

which is not the finite range distribution.
4. The conditional expectation,

\[ E[X_i | X_j = x_j] = \left[ p_j d(d+1)(p_1 - qx_j)^2 \right]^{-1} (1-p_j x_j) \]

\[ \left[ p_j d(p_1 - qx_j) + q - p_1 p_2 \right] \]  

(3.32)

and hence, the regression curves are non-linear.

When \( q=0 \), the equation (3.32) becomes,

\[ E[X_i | X_j = x_j] = \frac{1-p_i x_i}{(d+1)p_i} \]

and the regression turns out to be linear. In this case, the correlation coefficient is \(-\frac{1}{d}\), which is always negative.

5. The conditional distribution of \( X_i \) given \( X_j > x_j \) is

\[ h_i(x_i | X_j > x_j) = \frac{d(p_i - qx_j)}{1-p_j x_j} \]

(1- \( p_j x_j \)) \( (d-1) \)

and is of the finite range form. This fact enables the characterization of the bivariate finite range distribution (see Hitha and Nair (1991)).

6. The bivariate failure rate, defined in the equation (1.17) is

\[ h(x_1, x_2) = \left( \frac{d(p_1 - qx_2)}{(1-p_1 x_1 - p_2 x_2 + qx_1 x_2)} \right) \left( \frac{d(p_2 - qx_1)}{(1-p_1 x_1 - p_2 x_2 + qx_1 x_2)} \right) \]
Since the components $h_i(x_1, x_2)$ increases in $x_i$, the distribution (3.4) has increasing failure rate property.

7. The bivariate MRL vector, defined in the equation (1.24) is

$$r(x_1, x_2) = \left( \frac{1-p_1 x_1-p_2 x_2+q x_1 x_2}{(1+d)(p_1-q x_2)}, \frac{1-p_1 x_1-p_2 x_2+q x_1 x_2}{(d+1)(p_2-q x_1)} \right).$$

It is evident that $r_i(x_1, x_2)$ decreases as $x_i$ increases, and therefore, the distribution of $X=(X_1, X_2)$ is having DMRL-(2) (see Zahedi (1985)).

8. The relationship,

$$h_i(x_1, x_2) r_i(x_1, x_2) = k < 1$$

is a characteristic property of the model.

9. The Basu's failure rate defined in (1.13) is

$$a(x_1, x_2) = \frac{d[d(p_1-q x_2)(p_2-q x_1)+q-p_1 p_2]}{(1-p_1 x_1-p_2 x_2+q x_1 x_2)^2}.$$

10. The distribution has modified MRL (equations (1.33) and (1.34))

$$\left( \frac{1-p_1 x_1-p_2 x_2+q x_1 x_2}{(d+1)(p_1-q x_2)}, \frac{1-p_2 x_2}{p_2(d+1)} \right).$$
The three different models arising from the characterization theorem of Section 3.2 from a class that enjoy several interesting properties. Various distributional properties discussed in the previous sections form the basis of our future investigation in this direction. Regarding the Gumbel's exponential distribution, since a full discussion is available in Nair (1990), we mention only such properties that are of interest to the present study at the appropriate places.

A comparison of the AID and the parent distribution is informative in the study of the reliability characteristics. Some results using this approach in the univariate case have been reported in Gupta and Kirmani (1990). Since this concept does not appear to have been introduced in the bivariate case, the primary concern of the present chapter is to define bivariate AID. The failure rate (MRL) determines a distribution uniquely and therefore, it follows that the failure rate (MRL) of the AID will enable us to identify the form of AID. In situations where the failure rate (MRL) of the basic distribution has the same form as the corresponding characteristic of the AID, we can conclude that the parent distribution