CHAPTER FOUR

PERISHABLE INVENTORY PROBLEM WITH AGE-DEPENDENT REPLACEMENT POLICY

4.1 Introduction

In this chapter an inventory model of a single breed of chicken (white leghorn) is considered. The policy adopted is $(S,s)$ and lead time is assumed to be zero. Further shortage cost is infinity. Chicken are disposed of on attaining age $T$ (here 72 weeks). The life time of chicken are assumed to be independent and identically distributed random variables following exponential distribution with parameter $\lambda$. The demand process form a compound poisson process. The rate of arrival of demand is $\lambda$ per unit time. The quantity demanded at an epoch is independent of the quantity demanded at any other epoch and $q_i$ is probability that $i$ units $(i=1,2,\ldots)$ are demanded at a demand epoch. Since lead time is zero we may assume that the optimal 's' value is zero. The replenishment rate is assumed to be infinite. The time-dependent and also long run system state probabilities are calculated. The optimal 'S' value also is computed.
Single Commodity Inventory Problem has been analysed by several researchers. An account of the work in its initial stage can be had from Hadley and Whittin (1963) and Naddor (1966). Stochastic Inventory system is studied in depth by Arrow, Karlin and Scarf (1958). Sivazlian (1975) considers a single commodity inventory system with a demand forming a renewal process. Lead time is taken to be zero and no shortage is permitted. He obtains the limiting inventory level distribution as a discrete uniform and derives the optimal values of the ordering quantity. This is extended by Srinivasan (1979) to include lead time having arbitrary distribution function. Sahin (1979) considers an inventory problem with continuous state space and constant lead time. The binomial moments are computed in the case of an inventory problem with random lead time and demand taking place according to a compound renewal processes by Sahin (1983). An excellent review of perishable single commodity inventory problem is contained in Nahmias (1982). Kalpakam and Arivarignam (1985) deal with an inventory model with one exhibiting item having exponential life time distribution. They establish the limiting inventory level distribution. Krishnamoorthy and Lakshmi (1991) deal with an inventory
problem with Markov dependent demand quantities. This is especially useful in production inventory. Perishable inventory problems are also considered, among others by Manoharan and Krishnamoorthy (1989), and Krishnamoorthy, Narasimhalu and Iqbal Basha (1992).

4.2 Mathematical Modelling and analysis of the problem

Let \(0 < T_1 < T_2 \ldots < T_n < \ldots\) be the successive demand epochs. The successive replenishment epochs are identified as \(T_0', T_1', T_2', \ldots T_n' \ldots\). Note that the replenishment epochs need not coincide with a demand epoch since inventory level may fall to zero due to death of chicken. Further the successive replenishment epochs \(T_0', T_1', \ldots T_n' \ldots\) constitute a renewal process since at these epochs the inventory levels are brought back to \(S\).

The distribution of the time between two consecutive \(S\) to \(S\) transition is computed. This is then made use of compute the system state probabilities at any time (both finite and long run). The following notations are used:

\[
I(t) = \text{Number of birds alive at time } t; \ t \geq 0
\]
\[
P_n(t) = P\left\{I(t) = n / I(0) = S \right\}, \ n = s + 1, \ldots, S
\]
\[ \phi(u) = P \left\{ I(u) = m / I(0) = l \text{ without a demand epoch and no replenishment in the time interval } (0,u) \right\} \]

\[ \gamma_{\lambda, k}(u) \] denotes the gamma density with scale parameter \( \lambda \) and shape parameter \( k \).

Thus, \( \phi(u) \geq 0 \) for \( l \geq m \)
\[ \phi(u) = 0 \text{ otherwise} \]

\[ P_n(t) = p_n, \quad n=1,2,\ldots, S. \]

\[ t \to \infty \]

Obviously, \( O(u) \) stands for the probability that during an interval of duration \( u \), the number of deaths is \( m \).

Thus
\[ \phi(u) = \left( \frac{l}{m} \right) e^{\mu um} \left( 1 - e^{\mu u} \right)^{l-m} \]

While proceeding to compute \( P_n(t) \), for \( t > 0 \), note that up to time \( t \) there might have been none, one or more replenishments. These may happen with or without any demands in between. So the distribution of the time between two consecutive replenishments is computed first. There are three cases.

(i) No demand in between consecutive replenishment epochs and the inventory level falls from \( S \) to \( l \) (due to deaths) at the end of \( T \) time units from the
previous replenishments epoch. The remaining S - L birds are disposed off as their productive life has been completed on attaining age T.

The probability of this event is

\[
\mathcal{P}(T) = e^{-\lambda T} \sum_{S,L} \frac{1}{S!L!} (\lambda T)^{S+L} e^{-\lambda T}
\]

(ii) There are one or more demands between two replenishments. All the birds are either sold off and/or some of them died between these two epochs. Thus replenishment time (time between two replenishment epochs) is less than T in this case. The probability of this event is

\[
\sum_{k=1}^{S} \int_{u=0}^{U_k} \int_{l=0}^{L_k} \mathcal{P}(u_1) q_{m_1} \mathcal{P}(u_2) q_{m_2} \cdots
\]

Thus the distribution of any \( S = T \) is given by

\[
P[S = T] = \sum_{k=1}^{S} \int_{u=0}^{U_k} \int_{l=0}^{L_k} \mathcal{P}(u_1) q_{m_1} \cdots
\]

Here the factor \( \int_{S}^{T} \frac{1}{S!} \frac{(u-u_k)}{u_k - u_{k-1}} \) includes probability of left over, if any, dying before attaining age T.
There are one or more demands between two consecutive replenishments (time duration of this is T). Some birds are sold off and some die between these two epochs. The remaining are disposed off on attaining age T at which the next replenishment takes place.

The probability for this denoted by $H(x)$ equal to

$$
\sum_{k=1}^{S} \int_{u_k=0}^{T} \int_{u_k'=u_k}^{T} \sum_{l_1, l_2, \ldots, l_k \geq 0} \sum_{m_1, m_2, \ldots, m_k \geq 1} \sum_{l_1 + \ldots + l_k}
$$

Thus the distribution of any $Y_n = T_n - T_{n-1}$ is given by

$$
P[Y_n \leq x] = \text{expression (2.) for } x < T
$$

and $P[Y_n = T]$ is expression (i) + expression (iii)
Let the $n$-fold convolution ($n=1,2,\ldots$) of $H(x)$ be denoted by $H^n(x)$ and its density by $h(x)$ ($H(u)$ is defined to be identically equal to one).

Now the inventory level probabilities can be computed, at arbitrary (finite) time. For $t<T$,

$$P_s(t) = e^{-\lambda t} \phi_{S,S}(t) + \int_0^t \sum_{n=1}^\infty H^n(u) e^{-\lambda(t-u)} \phi_{S,S}(t-u) \, du$$

and for $n=S+1,\ldots,S-1$,

$$P_n(t) = \sum_{\left\{ l_1,\ldots,l_n \geq 0 \right\}} \phi_{S,S}(u_1) \ldots \phi_{S,S}(u_n)$$

For $t\geq T$,

$$P_s(t) = \int_t^\infty \sum_{m=1}^\infty h^m(u) e^{-\lambda(t-u)} \phi_{S,S}(t-u) \, du$$

and for $n$ satisfying $s+1 \leq n \leq S-1$,

$$P_n(t) = \sum_{\left\{ l_1,\ldots,l_n \geq 0 \right\}} h^m(u) \phi_{S,S}(u_1) \ldots \phi_{S,S}(u_n)$$
4.3 Limiting distribution

Now the limiting distribution of the system state can be computed. To this end $P_s(t)$ and $P_n(t)$ (given above by (4) and (5)) for $t > T$ are made use of. The Laplace transform of a function is defined by

$$\hat{f}(z) = \int_0^\infty e^{-zt}f(t)\,dt$$

Taking the Laplace transform on both sides of (4) and (5) we get

$${\mathcal P}_S(z) = \sum_{m=1}^\infty (\hat{f}(z))^m \frac{m!}{(\lambda+z)^{m+1}}$$

and for $n$ such that $1 \leq n \leq s-1$

$$\hat{P}_n(z) = \sum_{m=1}^\infty q_m \frac{1}{\lambda+z} \sum_{m=1}^{\infty} \frac{(\hat{f}(z))^m}{(\lambda+z)^{m+1}}$$

These can be inverted to obtain the required probabilities.

4.4 Optimisation problem

In this section the minimisation of total cost of running the system is discussed.

Let $C_1 = $ fixed cost of ordering

$C_2 = $ procurement cost per unit
$C_3 =$ holding cost per unit per unit time

$C_4 =$ loss due to death of a bird

$C_5 =$ loss due to disposal of the bird on attaining age $T$ if before that time it could not be sold off

The expected inventory (undecayed) held per unit time can be obtained from the inventory level distribution as given by (4) and (5). This provides the average holding cost per unit time. The average number of deaths is also obtained. Further the expected number of birds disposed off on attaining age $T$ can be calculated. These taken together provide the expression for the expected total cost incurred per unit time. The $S$ value that minimizes the total cost is easily obtained from this. It easily follows that the optimal re-ordering level is zero since lead time is zero and shortage cost is infinity.