CHAPTER III

SUPER EDGE-MAGIC GRAPHS

3.1 Introduction

The subject of edge-magic labelings of graphs had its origin in the work of Kotzig and Rosa [113,114] on what they called magic valuations of graphs. These labelings are currently referred to as either edge-magic labelings or edge-magic total labelings. These terms were coined by Ringel [149], and Wallis [194].

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It is known [149] that all caterpillars are super edge-magic. Cahit [51] showed that complete binary trees and 2-stars are super edge-magic. Figueroa-Centeno et al. [67] have proved that the galaxy \( mK_{1n} \) is super edge-magic for positive integers \( m \) and \( n \), \( n \) is odd. In [70], Figueroa-Centeno et al. showed that the forest \( P_m \cup K_{1n} \) is super edge-magic for every positive integer \( m \geq 4 \) and \( n \geq 1 \). In 1996, Ringel and Llado [149] conjectured that all trees are edge-magic. In [63], Enomoto et al. made a stronger conjecture that every tree is super edge-magic (both conjectures still remain open). J.C. Bermond [38] conjectured that all lobsters are graceful. In [7] it is given that lobsters are arithmetic.
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3.1 Introduction

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It is known [149] that all caterpillars are super edge-magic. Cahit [51] showed that complete binary trees and 2-stars are super edge-magic. Figueroa-Centeno et al. [67] have proved that the galaxy $mK_{1,n}$ is super edge-magic for positive integers $m$ and $n$, $m$ is odd. In [70], Figueroa-Centeno et al. showed that the forest $P_m \cup K_{1,n}$ is super edge-magic for every positive integer $m \geq 4$ and $n \geq 1$. In 1996, Ringel and Llado [149] conjectured that all trees are edge-magic. In [63], Enomoto et al. made a stronger conjecture that every tree is super edge-magic (both conjectures still remain open). J.C. Bermond [38] conjectured that all lobsters are graceful. In [7] it is given that lobsters are arithmetic.
Figueroa-Centeno et al. [71] proved the following: if \( G \) is a (super) edge-magic 2-regular graph, then \( G \odot \overline{K}_n \) is (super) edge-magic for every positive integer \( n \); and the \( n \)-crown \( C_m \odot \overline{K}_n \) is super edge-magic for every two integers \( m \geq 3 \) and \( n \geq 1 \). Yegnanarayanan [203] showed that the graph obtained by introducing \( n \) new pendant edges at each vertex of the outermost \( C_3 \) in \( P_i \times C_3 \) is super edge-magic for \( t \geq 2 \). In [66] Figueroa-Centeno et al. proved that the ladder \( L_n \equiv P_n \times K_2 \) is super edge-magic for odd \( n \) and the generalized prism \( C_m \times P_n \) is super edge-magic if \( m \) is odd and \( n \geq 2 \). These two results were independently obtained by Tsuchiya and Yokomura [193] also.

Kathiresan [109] showed that \( L_n \odot K_1 \) is graceful.

Balakrishnan and Sampathkumar [28] have proved that the total graph \( T(P_n) \) is harmonious. Balakrishnan, Selvam and Yegnanarayanan [30] showed that \( T(P_n) \) is elegant.

In [63] it is proved that the complete bipartite graph \( K_{m,n} \) is super edge-magic if and only if \( m = 1 \) or \( n = 1 \). Balakrishnan et al. [28] proved that the graph \( \overline{K}_n + 2K_2 \) is magic if and only if \( n = 3 \) and harmonious if and only if \( n \) is even. Helms have been shown to be harmonious [160].

Delorme, Maheo, Thuillier, Koh and Teo [60] proved that any cycle with a chord is graceful. Xu [200] showed that all cycles with a
chord are harmonious except for $C_6$ in the case where the distance in $C_6$ between the end-vertices of the chord is 2.

Truszczyński [192] studied unicyclic graphs and proved that several classes of such graphs are graceful. He conjectured that all unicyclic graphs except $C_n$ when $n = 1$ or $2 \pmod{4}$ are graceful. In 1996, Arumugam and Germina [10] have shown that all unicyclic graphs are indexable.

Figueroa-Centeno et al. [66,72] studied the relation between super edge-magic labelings and other classes of labelings such as sequential, harmonious, cordial, graceful and felicitous.

In this chapter we obtain some new classes of super edge-magic graphs and study the relation between super edge-magic labeling and other classes of labelings.

### 3.2 New classes of super edge-magic graphs

The following results on trees give support to the conjecture [63] that all trees are super edge-magic.

**Theorem 3.2.1** The 3-star $S_{m,3}$ is super edge-magic if $m$ is odd.

**Proof.** Suppose that $m$ is odd. Let $x$ denote the vertex of degree $m$ in $S_{m,3}$ and $xu_1v_1w_i$ denote the $i$th path of length 3 for $1 \leq i \leq m$.

Since $S_{1,3} \cong P_4$ and paths are super edge-magic, the result is true when $m = 1$. Now suppose that $m$ is odd and $m \geq 3$. Let $n = 3m + 1$. 

Consider the vertex labeling

\[ f: V(S_{m,3}) \rightarrow \{1, 2, 3, \ldots, n\} \]

such that

\[ f(x) = \frac{n+2}{3} \]
\[ f(u_1) = n \]

\[ f(u_{2i}) = 2i \quad \text{for } 1 \leq i \leq \frac{n-4}{6} \]
\[ f(u_{2i+1}) = \frac{n+5}{3} - 2i - 1 \quad \text{for } 1 \leq i \leq \frac{n-4}{6} \]
\[ f(v_{2i}) = \frac{n+5}{3} + 2i - 2 \quad \text{for } 1 \leq i \leq \frac{n-4}{6} \]
\[ f(v_{2i+1}) = 2i + 1 \quad \text{for } 0 \leq i \leq \frac{n-4}{6} \]
\[ f(w_1) = \frac{2n+1}{3} \]
\[ f(w_{2i}) = \frac{5n-2}{6} - i + 1 \quad \text{for } 1 \leq i \leq \frac{n-4}{6} \]
\[ f(w_{2i+1}) = n - i \quad \text{for } 1 \leq i \leq \frac{n-4}{6} \]

Note that \( S = \{ f(x) + f(y) : xy \in E(S_{m,3}), m \geq 3 \text{ is odd} \} \)
\[ = \{ \frac{n+8}{3}, \frac{n+11}{3}, \ldots, \frac{4n+2}{3} \} \]

is a set of \( n-1 \) consecutive integers.
Hence, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( S_{m,3} \) with valence \( k = p + q + s = n + n - 1 + \frac{n + 8}{3} = \frac{7n + 5}{3} \) when \( m \geq 3 \) is odd.

**Example** Figure 3.2.1 shows the super edge-magic graph \( S_{7,3} \).
Definition 3.2.1 Let $T_1$ be a caterpillar obtained by putting one end-vertex at each vertex $a_i$ of the path $P_n$, $1 \leq i \leq n$. Let $T$ be the lobster formed by joining a copy of $P_2$ at each end-vertex $b_i$ of $T_1$, $1 \leq i \leq n$.

Figure 3.2.2 shows a lobster obtained using the above construction.

Theorem 3.2.2 The lobster $T$ defined above is super edge-magic for all integers $n \geq 3$.

Proof. We consider two cases.

Case 1 $n$ is even
Let $c_i$ denote the end-vertex of $T$ at $b_i$, $1 \leq i \leq n$.

Define a vertex labeling $f: V(T) \to \{1, 2, 3, \ldots, 3n\}$ such that

$$f(a_i) = \begin{cases} 2n+i & \text{if } i \text{ is odd, } 1 \leq i \leq n \\ 2n+i & \text{if } i \text{ is even, } 1 \leq i \leq n \end{cases}$$

Note that $S = (f(a_i) + f(c_i) : xy \in E(T))$, $n$ is even, $n \geq 3$.

Therefore, $S = (2n+2, 2n+1, 2n, 2n-1, \ldots, 3n-4)$.

Example. Figure 3.2.3 shows the super edge-magic labeling of the lobster with $n = 6$.
Super edge-magic graphs

\[ f(a_i) = \begin{cases} 
  i & \text{if } i \text{ is even}, 1 \leq i \leq n \\
  2n + i & \text{if } i \text{ is odd}, 1 \leq i \leq n 
\end{cases} \]

\[ f(b_i) = \begin{cases} 
  i & \text{if } i \text{ is odd}, 1 \leq i \leq n \\
  2n + i & \text{if } i \text{ is even}, 1 \leq i \leq n 
\end{cases} \]

\[ f(c_1) = 2n \]

\[ f(c_{2i+3}) = \frac{3n}{2} (1 + i) \text{ for } 0 \leq i \leq \frac{n-4}{2} \]

\[ f(c_{n-2i}) = \frac{3n}{2} + i \text{ for } 0 \leq i \leq \frac{n-2}{2} \]

Note that \( S = \{ f(x) + f(y) : xy \in E(T), \text{n is even}, n \geq 3 \} \)
\[ = \{ 2n-2, 2n-1, 2n, 2n+1, \ldots, 5n-4 \} \]

Therefore, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( T \) with valence \( k = p + q + s = 8n-3 \).

**Example** Figure 3.2.3 shows the super edge-magic labeling of the lobster \( T \) with \( n = 8 \).

![Figure 3.2.3](image-url)
Case 2. \( n \) is odd

Consider the vertex labeling \( f: V(T) \to \{1, 2, 3, \ldots, 3n\} \) where

\[
\begin{align*}
  f(a_i) &= \begin{cases} 
    i & \text{if } i \text{ is even, } 1 \leq i \leq n \\
    n+i & \text{if } i \text{ is odd, } 1 \leq i \leq n
  \end{cases} \\
  f(b_i) &= \begin{cases} 
    i & \text{if } i \text{ is odd, } 1 \leq i \leq n \\
    n+i & \text{if } i \text{ is even, } 1 \leq i \leq n
  \end{cases}
\end{align*}
\]

\( f(c_i) = 3n \)

\( f(c_{2i+1}) = 3n - i \) for \( 1 \leq i \leq \frac{n-1}{2} \); \( f(c_{n+2i+1}) = 2n + i \) for \( 1 \leq i \leq \frac{n-1}{2} \)

Since \( S = \{ f(x) + f(y) : xy \in E(T) \text{, } n \text{ is odd, } n \geq 3 \} \)

\( = \{ n+2, n+3, \ldots, 4n \} \),

by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( T \) with valence \( k = p + q + s = 7n + 1 \).

**Example** Figure 3.2.4 shows the super edge-magic labeling of the lobster \( T \) with \( n = 5 \).

![Figure 3.2.4](image-url)
**Definition 3.2.2** Let \( \{ \alpha_1 K_{1,n_1}, \alpha_2 K_{1,n_2}, \ldots, \alpha_p K_{1,n_p} \} \) be a family of stars where \( \alpha_i K_{1,n_i} \) denotes \( \alpha_i \) disjoint isomorphic copies of \( K_{1,n_i} \) for \( 1 \leq i \leq p \) and \( \alpha_i \geq 1 \). Let \( H_{ij} \) be the \( j \)th isomorphic copy of \( K_{1,n_i} \) and \( v_{ijk} \) be the end-vertices of \( H_{ij} \), \( k = 1, 2, \ldots, n_i \). Adjoin a new vertex \( w \) which is adjacent with one end-vertex of each star. The tree thus obtained is denoted by \( H_w^{(\alpha_1 + \alpha_2 + \cdots + \alpha_p)} \). Trees of this kind are referred to as banana trees, by some authors.

**Theorem 3.2.3** The banana tree \( H_w^{(\alpha_1 + \alpha_2 + \cdots + \alpha_p)} \)
corresponding to the family of stars \( \{ \alpha_1 K_{1,n_1}, \alpha_2 K_{1,n_2}, \ldots, \alpha_p K_{1,n_p} \} \),
\( 1 \leq n_1 < n_2 < \cdots < n_p, p \geq 2 \) and \( \alpha_1 + \alpha_2 + \cdots + \alpha_i \leq n_i \), \( i = 1, 2, \ldots, p \),
is super edge-magic.

**Proof.** Consider the family of stars \( \{ \alpha_1 K_{1,n_1}, \alpha_2 K_{1,n_2}, \ldots, \alpha_p K_{1,n_p} \} \). Let \( H_{ij} \) be the \( j \)th isomorphic copy of \( K_{1,n_i} \), \( i = 1, 2, \ldots, p \). Let \( v_{ijk} \) be the end-vertices of \( H_{ij} \), \( k = 1, 2, \ldots, n_i \) and \( u_{ij} \) be the center of \( H_{ij} \).
Let \( w \) be the new vertex adjacent to one end-vertex \( v_{ij} \) from each star \( H_{ij} \) of the family where \( \beta_{ij} = \alpha_0 + \alpha_1 + \cdots + \alpha_{i-1} + j \) and \( \alpha_0 = 0 \). The new tree obtained is denoted by \( H_w^{(\alpha_1 + \alpha_2 + \cdots + \alpha_p)} \)
and has \( \alpha_1(n_1+1) + \alpha_2(n_2+1) + \cdots + \alpha_p(n_p+1) + 1 \) vertices and \( \alpha_1n_1 + \alpha_2n_2 + \cdots + \alpha_p n_p + (\alpha_1 + \alpha_2 + \cdots + \alpha_p) \) edges.
Let \( p_1 = \alpha_1(n_1 + 1) + \alpha_2(n_2 + 1) + \cdots + \alpha_p(n_p + 1) + 1 \). Define a vertex labeling \( f : V(H_w^{(a_1 + a_2 + \cdots + a_p)}) \to \{ 1, 2, \ldots, p_1 \} \) such that

\[
\begin{align*}
 f(v_{ijk}) &= (j - 1)n_1 + k \quad \text{for } 1 \leq j \leq \alpha_1, 1 \leq k \leq n_1. \\
 f(v_{ijk}) &= f(v_{i-1a_i-1n_i-1}) + (j - 1)n_i + k \quad \text{for } 2 \leq i \leq p, 1 \leq j \leq \alpha_i, 1 \leq k \leq n_i. \\
 f(w) &= f(v_{p\alpha_pn_p}) + 1. \\
 f(u_{ij}) &= f(w) + (\alpha_0 + \alpha_1 + \cdots + \alpha_{i-1} + j), \quad 1 \leq i \leq p, 1 \leq j \leq \alpha_i.
\end{align*}
\]

Note that

\[
S = \{ \alpha_1n_1 + \alpha_2n_2 + \cdots + \alpha_pn_p + 2, \alpha_1n_1 + \alpha_2n_2 + \cdots + \alpha_pn_p + 3, \ldots, 2(\alpha_1n_1 + \alpha_2n_2 + \cdots + \alpha_pn_p) + (\alpha_1 + \alpha_2 + \cdots + \alpha_p) + 1 \}.
\]

Hence, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( H_w^{(a_1 + a_2 + \cdots + a_p)} \) with valence

\[
k = p + q + s = 3(\alpha_1n_1 + \alpha_2n_2 + \cdots + \alpha_pn_p) + 2(\alpha_1 + \alpha_2 + \cdots + \alpha_p) + 3. \quad \Box
\]

**Example** Figure 3.2.5 shows the graph \( H_w^{(2 + 2 + 1)} \) and Figure 3.2.6 shows the super edge-magic labeling of the graph \( H_w^{(2 + 2 + 1)} \).
Super edge-magic graphs

Figure 3.2.5
Super edge-magic graphs

Definition 3.2.3 A graph $G(t, m) = P_{t} \coprod C_{2m - 1}$ where $=$
stands for the path on $t$ vertices $(t \geq 2)$ with an
odd cycle. Obtain $G(t, m, n)$ by introducing a new
pendant edges at each of the outermost odd cycle
in $G(t, m)$.

Theorem 3.2.4 For $n \geq 2$ the graph $G(t, m, n)$ is super
edge-magic.

Proof. Let $v_{1}$ be a vertex of the innermost $C_{2m - 1}$ and $v_{2}$ be
the other vertices of the cycle taken in the clockwise
direction. For $2 \leq i \leq t$, let $v_{2i}$ be the vertex of the $i^{th}$ copy of
$C_{2m - 1}$ adjacent to the vertex $v_{1}$ in the $(i - 1)^{th}$ copy
$C_{2m - 1}$ and take the other vertex $v_{2i}$ as the clockwise vertex of the first
copy of $C_{2m - 1}$ adjacent to the vertex $v_{1}$ of the outermost cycle.

Consider the vertex labeling

\[ f: V(G(t, m, n)) \rightarrow \{1, 2, \ldots, (t+1)(n+1)\} \]

such that

\[ f(v_{1}) = 1 \]

\[ f(v_{2i}) = \begin{cases} \frac{j+2}{2} & \text{if } j \text{ is odd} \\ \frac{j-2}{2} & \text{if } j \text{ is even} \end{cases} \]

for $1 \leq i \leq t$ and $1 \leq j \leq (2m - 1)$.

Figure 3.2.6
**Definition 3.2.3** Consider the graph $G(t,m) = P_t \times C_{2m+1}$ where $\times$ stands for the cartesian product of a path on $t$ vertices ($t \geq 2$) with an odd cycle. Obtain a new graph $G(t,m,n)$ by introducing $n$ new pendant edges at each vertex of the outermost odd cycle in $G(t,m)$.

**Theorem 3.2.4** For $t \geq 2$ and $m \geq 2$ the graph $G(t,m,n)$ is super edge-magic.

**Proof.** Let $v_{11}$ be any fixed vertex of the innermost $C_{2m+1}$ and $v_{12}, v_{13}, \ldots, v_{1(2m+1)}$ be the other vertices of the cycle taken in the clockwise direction. For $2 \leq i \leq t$, let $v_{i1}$ be the vertex of the $i^{th}$ copy of $C_{2m+1}$ adjacent to the vertex $v_{(i-1)(2m+1)}$ in the $(i-1)^{th}$ copy of $C_{2m+1}$ and take the other vertices $v_{ij}$ in the clockwise direction as in the first copy of $C_{2m+1}$. Let $v_{tjk}$ denote the pendant vertex adjacent to the vertex $v_{tj}$ of the outermost $C_{2m+1}$ for $1 \leq k \leq n$ and $1 \leq j \leq (2m+1)$.

Consider the vertex labeling

$$f: V(G(t,m,n)) \rightarrow \{1, 2, \ldots, (2m+1)(t+n)\}$$

such that

$$f(v_{ij}) = \begin{cases} 
(i-1)(2m+1) + \frac{j+1}{2} & \text{if } j \text{ is odd} \\
(i-1)(2m+1) + m + \frac{j+2}{2} & \text{if } j \text{ is even}
\end{cases}$$

for $1 \leq i \leq t$ and $1 \leq j \leq (2m+1)$ and
\[ f(u_{jk}) = (2m + 1) (t + k - 1) + (2m + 2 - j) \text{ for } 1 \leq j \leq (2m + 1) \text{ and } 1 \leq k \leq n. \]

Note that \( S = \{ f(x) + f(y) : xy \in E(G(t,m,n)), t \geq 2, m \geq 2 \} \)
\[ = \{ m + 2, m + 3, \ldots, (m + 1) + (2m + 1) (2t + n - 1) \} \]

is a set of consecutive integers and hence, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( G(t,m,n) \) with valence \( k = p + q + s = (2m + 1) (3t + 2n - 1) + m + 2 \), for \( t \geq 2 \) and \( m \geq 2 \).

**Example** Figure 3.2.7 shows the super edge-magic labeling of the graph \( G(3,2,2) \).

\[ f(v) = \begin{cases} \frac{n+1}{2} & \text{if } i \text{ is even} \\ m + \frac{i}{2} & \text{if } i \text{ is odd} \end{cases} \]

\[ f(u_{jk}) = \begin{cases} \frac{15n+2}{2} & \text{if } i \text{ is even} \\ 16m + 4n + 1 & \text{if } i \text{ is odd} \end{cases} \]

**Figure 3.2.7**
Theorem 3.2.5 The graph $C_n \odot P_2$ is super edge-magic for all odd $n \geq 3$.

Proof. Let $n$ be an odd integer and $n = 2m + 1 \geq 3$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle $C_n$. Now $C_n \odot P_2$ is the graph obtained by attaching $P_2$ to each vertex of $C_n$. Let $a_i, b_i, 1 \leq i \leq n$ be the vertices adjacent to the rim vertices $v_i$ of $C_n$ in $C_n \odot P_2$. The graph $C_n \odot P_2$ has $3n$ vertices and $4n$ edges. Define a vertex labeling

$$f: V(C_n \odot P_2) \to \{1, 2, \ldots, 3n\}$$

such that

$$f(v_i) = \begin{cases} 
\frac{i+1}{2} & \text{if } i \text{ is odd} \\
 2n + m + \frac{i+2}{2} & \text{if } i \text{ is even}
\end{cases}$$

$$f(a_i) = 2n + 1 - i \quad \text{for } 1 \leq i \leq n$$

$$f(b_{2i}) = 2n + i \quad \text{for } 1 \leq i \leq m$$

$$f(b_1) = 2n + m + 1$$

$$f(b_{2i+1}) = 2n + m + 1 + i \quad \text{for } 1 \leq i \leq m$$

It is easy to see that

$$S = \{f(x) + f(y) : xy \in E(C_n \odot P_2)\} = \{m + 2, m + 3, \ldots, m + 4n + 1\}$$

is a set of $4n$ consecutive integers. Hence, by Lemma 1.2.2, $f$ extends to a super edge-magic labeling of $C_n \odot P_2$ with valence

$$k = p + q + s = \frac{15n + 3}{2}, \text{ when } n \geq 3 \text{ is odd.} \quad \square$$
Example Figure 3.2.8 shows the super edge-magic labeling of the graph $C_s \odot P_2$.

Theorem 3.2.6 The graph $C_n \odot P_3$ is super edge-magic for all odd $n \geq 3$.

Proof. Let $C_n$ be an odd cycle with $n = 2m + 1 \geq 3$ vertices. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle $C_n$. Let $P_3$ be a path on three vertices. Now $C_n \odot P_3$ is the graph obtained by attaching $P_3$ to each vertex of $C_n$ and it has $4n$ vertices and $6n$ edges.
Define a vertex labeling $f: V(C_n \odot P_3) \to \{1, 2, \ldots, 4n\}$ such that

$$f(v_i) = \begin{cases} 
\frac{i+1}{2} & \text{if } i \text{ is odd} \\
\frac{m+\frac{i+2}{2}}{2} & \text{if } i \text{ is even}
\end{cases}$$

Since $n$ is odd, there are exactly $m$ even $f$-values and $m + 1$ odd $f$-values for the rim vertices. Let us label the $3n$ vertices outside the rim of $C_n$ in $C_n \odot P_3$ as follows. Let $u_1, u_2, \ldots, u_m$ be the vertices of degree two outside the rim, adjacent to the rim vertices whose $f$-values are $2m, 2m - 2, \ldots, 4, 2$ respectively. Again let $u_{n+1}, u_{n+2}, \ldots, u_{n+m}$ be the remaining vertices of degree two, adjacent to the rim vertices whose $f$-values are $2m, 2m - 2, \ldots, 4, 2$ respectively.

Let $u_{m+1}, u_{m+2}, \ldots, u_n$ be the vertices of degree two outside the rim, adjacent to the rim vertices whose $f$-values are $n, n - 2, \ldots, 3, 1$ respectively. Also let $u_{n+m+1}, u_{n+m+2}, \ldots, u_{2n}$ be the remaining vertices of degree two outside the rim, adjacent to the rim vertices whose $f$-values are $n, n - 2, \ldots, 3, 1$ respectively. Let $u_{2n+1}, u_{2n+2}, \ldots, u_{2n+m+1}$ be the vertices of degree three outside the rim adjacent to the rim vertices whose $f$-values are $n, n - 2, \ldots, 3, 1$ respectively. Finally, let $u_{2n+m+2}, u_{2n+m+3}, \ldots, u_{3n}$ be the vertices of degree three adjacent to the rim vertices whose $f$-values are $2m, 2m - 2, \ldots, 4, 2$ respectively.

Now define $f(u_i) = n + i$ for $1 \leq i \leq 3n$. 
Note that \( S = \{ f(x) + f(y) : xy \in E(C_n \odot P_3) \} = \{ m + 2, m + 3, \ldots, m + 6n + 1 \} \) is a set of consecutive integers.

Hence, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( C_n \odot P_3 \) with valence \( k = p + q + s = \frac{21n + 3}{2} \).

**Example** Figure 3.2.9 shows the super edge-magic labeling of the graph \( C_7 \odot P_3 \).

**Figure 3.2.9**
Definition 3.2.4 Let $L_n$ denote the ladder graph $P_n \times P_2$ and $L_n \odot K_1$ be the graph obtained by adjoining an edge at each vertex of $L_n$.

Theorem 3.2.7 The graph $L_n \odot K_1$ is super edge-magic for odd $n$.

Proof. Let $V(L_n) = \{ u_1, u_2, \ldots, u_n; v_1, v_2, \ldots, v_n \}$ and $E(L_n) = \{ u_iu_{i+1}, v_iv_{i+1}, u_jv_j, 1 \leq i \leq (n-1), 1 \leq j \leq n \}$.

Let $u^1_i$ and $v^1_i$ be the vertices adjacent to $u_i$ and $v_i$ respectively in $L_n \odot K_1$. Then $V(L_n \odot K_1) = \{ u_i, v_i, u^1_i, v^1_i : 1 \leq i \leq n \}$ and $E(L_n \odot K_1) = \{ u_iu_{i+1}, v_iv_{i+1}, u_jv_j, u^1_jv^1_j, 1 \leq i \leq (n-1), 1 \leq j \leq n \}$.

The graph $L_n \odot K_1$ has $4n$ vertices and $5n - 2$ edges. Consider the vertex labeling $f: V(L_n \odot K_1) \to \{ 1, 2, \ldots, 4n \}$ where
A super edge-magic labeling of $L_n \otimes K_1$ is given by the function $f(x)$:

$$f(x) = \begin{cases} \frac{4n+i+1}{2} & \text{if } x = u_i, \text{ } i \text{ odd and } 1 \leq i \leq n \\ \frac{5n+i+1}{2} & \text{if } x = u_i, \text{ } i \text{ even and } 1 \leq i \leq n \\ \frac{3n+i}{2} & \text{if } x = v_i, \text{ } i \text{ odd and } 1 \leq i \leq n \\ \frac{2n+i}{2} & \text{if } x = v_i, \text{ } i \text{ even and } 1 \leq i \leq n \\ n & \text{if } x = v_1^1 \\ \frac{7n+1}{2} & \text{if } x = v_2^1 \\ i & \text{if } x = u_{2i+1}^1, \text{ } 1 \leq i \leq \left(\frac{n-1}{2}\right) \\ \frac{n+2i-1}{2} & \text{if } x = v_{2i}^1, \text{ } 2 \leq i \leq \left(\frac{n-1}{2}\right) \\ \frac{7n+2i+1}{2} & \text{if } x = u_{2i-1}^1, \text{ } 1 \leq i \leq \left(\frac{n-1}{2}\right) \\ \frac{3n+1+i}{2} & \text{if } x = u_{2i}^1, \text{ } 1 \leq i \leq \left(\frac{n-3}{2}\right) \\ \frac{n+1}{2} & \text{if } x = u_{n-1}^1 \\ 3n+1 & \text{if } x = u_n^1 \end{cases}$$

Note that

$$S = \{ f(x) + f(y) : xy \in E(L_n \otimes K_1) \} = \left\{ \frac{3n+5}{2}, \frac{3n+7}{2}, ..., \frac{13n-1}{2} \right\}$$

is a set of consecutive integers. Hence, by Lemma 1.2.2, $f$ extends to a super edge-magic labeling of $L_n \otimes K_1$ with valence.
k = \frac{21n + 1}{2}, \text{ for all odd } n.

**Example** Figure 3.2.10 gives the super edge-magic labeling of \( L_5 \odot K_1 \).

**Proof.** Let \( u_i \) denote the path \( u_j, e_j \) for all \( j \) such that \( u_j, e_j \) for \( 1 \leq j \leq (n - 1) \). The vertex set and edge set of \( T(P_n) \) are defined as follows:

\[
V(T(P_n)) = \{ u_i, e_j : 1 \leq i \leq (n - 1) \}.
\]

\[
E(T(P_n)) = \{ u_i, e_j, u_i, e_j, e_j, e_j : 1 \leq i \leq (n - 1) \}.
\]

Note that \( T(P_n) \) has \( q = 2p - 3 \). Now consider the vertex labeling:

\[
f : V(T(P_n)) \to \{ 1, 2, \ldots, (2n - 1) \}
\]

such that \( f(u_i) = 2i - 1 \) for \( 1 \leq i \leq (n - 1) \) and \( f(e_j) = 2j \) for \( 1 \leq j \leq (n - 1) \).

It follows that \( S = \{ f(x) + f(xy) : e_j \in E(T(P_n)) \} = \{ 3, 4, \ldots, (4n - 3) \} \) is a set of consecutive integers. Therefore, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( T(P_n) \). \qed

**Example** Figure 3.2.11 shows the super edge-magic labeling of the graph \( T(P_n) \).

By Lemma 1.2.1, if \( G \) is a super edge-magic \( (p, q) \)-graph then

\[ q \leq 2p - 3. \]

Next theorem gives a super edge-magic graph with \( q = 2p - 3 \).
**Theorem 3.2.8** The total graph $T(P_n)$ is super edge-magic for any positive integer $n$.

**Proof.** Let $P_n$ denote the path $u_1u_2\cdots u_n$ and $e_j$ denote the edge $u_ju_{j+1}$ for $1 \leq j \leq (n-1)$. The vertex set and edge set of $T(P_n)$ are defined as follows.

$$V(T(P_n)) = \{ u_i, e_j : 1 \leq i \leq n, 1 \leq j \leq (n-1) \}$$ and

$$E(T(P_n)) = \{ u_iu_{i+1}, e_je_{j+1}, u_ie_i, e_{i-1}u_{i+1} : 1 \leq i \leq (n-1), 1 \leq j \leq (n-2) \}.$$ 

Note that $T(P_n)$ has $2n-1$ vertices and $4n-5$ edges so that $q = 2p - 3$.

Now consider the vertex labeling $f : V(T(P_n)) \rightarrow \{ 1, 2, \ldots, (2n-1) \}$ such that

$$f(u_i) = 2i - 1 \text{ for } 1 \leq i \leq n, \quad f(e_i) = 2i \text{ for } 1 \leq i \leq (n-1).$$

It follows that $S = \{ f(x) + f(y) : xy \in E(T(P_n)) \} = \{ 3, 4, \ldots, (4n-3) \}$ is a set of consecutive integers. Therefore, by Lemma 1.2.2, $f$ extends to a super edge-magic labeling of $T(P_n)$ with valence $k = 6n - 3$. \qed

**Example** Figure 3.2.11 shows the super edge-magic labeling of the graph $T(P_4)$. 

![Figure 3.2.11](image-url)
Theorem 3.2.9 The graph $K_n + 2K_2$ is not super edge-magic.

Proof. The graph $K_n + 2K_2$ has $n + 4$ vertices and $4n + 2$ edges. Thus, $p = n + 4$, $q = 4n + 2$, $2p - 3 = 2n + 5$ and $q > 2p - 3$ if $n \geq 2$. Hence by Lemma 1.2.1, $K_n + 2K_2$ is not super edge-magic if $n \geq 2$.

Note that the graph $K_1 + 2K_2 = T_2$, the friendship graph with two triangles. But by Lemma 1.2.3, the friendship graph with $m$ triangles, $T_m$ is super edge-magic if and only if $3 \leq m \leq 5$ and $m = 7$. Hence $K_1 + 2K_2$ is not super edge-magic. This completes the proof of the theorem. □

Theorem 3.2.10 The graph consisting of a cycle $C_n$ with a chord joining two vertices at a distance 3 is super edge-magic for all odd $n$, $n \geq 7$.

Proof. Let $G$ be the graph consisting of a cycle $C_n$ with a chord joining two vertices of $C_n$ ($n \geq 7$) at a distance 3.

Let $V(G) = \{ v_1, v_2, \ldots, v_n \}$. Next join the vertices $v_1$ and $v_{n-2}$ as a chord for $G$ so that $d(v_1, v_{n-2}) = 3$. Note that $G$ has $n$ vertices and $n + 1$ edges. Consider the vertex labeling $f : V(G) \rightarrow \{ 1, 2, \ldots, n \}$ such that

$$f(v_i) = \begin{cases} 
\frac{i + 1}{2} & \text{if } i \text{ is odd} \\
n + i + 1 & \text{if } i \text{ is even}
\end{cases}$$
Note that

\[ S = \{ f(x) + f(y) : xy \in E(G) \} = \{ \frac{n+1}{2}, \frac{n+3}{2}, \ldots, \frac{3n+1}{2} \} \]

is a set of consecutive integers.

Hence, by Lemma 1.2.2, \( f \) is a super edge-magic labeling of \( G \) with valence \( k = p + q + s = \frac{5n+3}{2} \).

**Example** Figure 3.2.12 shows the super edge-magic labeling of the cycle \( C_7 \) with a chord.

\[ s_i = \min \{ S_i \}, \text{ where } S_i = \{ f_i(x) + f_i(y) : xy \in E(G_i) \}. \]

It is easy to see that \( s_i, s_{i+1}, \ldots, s_{i+q}, \ldots \) are \( q \) consecutive integers.

Hence, by Lemma 1.2.2, we extend to a super edge-magic labeling of \( G \) with valence \( k = q + s_i \).

**Example** Figures 3.2.13 and 3.2.14 show the super edge-magic labeling of \( G_1 \) and \( G \) respectively.

**Theorem 3.2.11** Let \( G_1 \) be a super edge-magic unicyclic graph and \( G_2 = K_{1,n} \) with \( V(K_{1,n}) = \{ v, u_1, u_2, \ldots, u_n \} \) and \( E(K_{1,n}) = \{ vu_i \mid 1 \leq i \leq n \} \).

Let \( G \) be the graph obtained by joining \( u_1 \) to all the vertices of \( G_1 \).

Then \( G \) is a super edge-magic graph.
**Proof.** Suppose that $G_1$ has $p_1$ vertices and $q_1$ edges. Let $f_1: V(G_1) \to \{1, 2, \ldots, p_1\}$ be a super edge-magic labeling of $G_1$.

Note that $G$ is a $(p, q)$ graph with $p = p_1 + n + 1$ and $q = q_1 + p_1 + n$.

Define a vertex labeling $f: V(G) \to \{1, 2, \ldots, p\}$ such that

- $f(x_i) = f_1(x_i)$ if $x_i \in V(G_1)$
- $f(v) = p_1 + 1$
- $f(u_i) = p_1 + 1 + i$, $1 \leq i \leq n$.

Let $s_i = \min(S_i)$ where $S_i = \{f_1(x) + f_1(y) : x y \in E(G_1)\}$.

It is easy to see that $S = \{f(x) + f(y) : x y \in E(G)\} = \{s_1, s_1 + 1, \ldots, s_1 + q_1, \ldots, s_1 + q - 1\}$ is a set of $q$ consecutive integers.

Hence, by Lemma 1.2.2, $f$ extends to a super edge-magic labeling of $G$ with valence $k = p + q + s_1$.

**Example** Figure 3.2.13 and Figure 3.2.14 show the super edge-magic labeling of $G_1$ and $G$ respectively.

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**Theorem 3.2.12** Let $G_1, G_2, \ldots, G_m$ be $m$ disjoint $n$-cycles having vertex sets $V_i = \{v_i^1, v_i^2, \ldots, v_i^n\}$, $1 \leq i \leq m$, where $n$ is odd and $n \geq 3$. Let $G$ be the graph obtained by joining $v_i^1$ to $v_j^1$, $1 \leq j \leq n$ and $v_i^k$ to $v_j^{k+1}$, $1 \leq k \leq n - 1$. Then $G$ is a super edge-magic graph.

**Proof.** The graph $G$ has $mn$ vertices and $n(2m - 1)$ edges.
Theorem 3.2.12 Let $G_1, G_2, \ldots, G_m$ be $m$ disjoint $n$-cycles having vertex sets $V_i = \{ v_1^i, v_2^i, \ldots, v_n^i \}$, $i = 1, 2, \ldots, m$, where $n$ is odd and $n \geq 3$. Let $G$ be the graph obtained by joining $v_n^1$ to $v_j^2$, $1 \leq j \leq n$ and $v_n^k$ to $v_j^{k+1}$, $1 \leq j \leq n$, $2 \leq k \leq (m - 1)$. Then $G$ is a super edge-magic graph.

Proof. The graph $G$ has $mn$ vertices and $n(2m-1)$ edges.
Define a vertex labeling \( f: V(G) \rightarrow \{1, 2, ..., mn\} \) such that

\[
f(v_i) = \begin{cases} 
\frac{i+1}{2} & \text{if } i \text{ is odd, } 1 \leq i \leq n \\
\frac{n+1+i}{2} & \text{if } i \text{ is even, } 1 \leq i \leq n 
\end{cases}
\]

\[
f(v_r') = 5r-4 \quad \text{if } 2 \leq r \leq m
\]

\[
f(v_i') = \begin{cases} 
f(v_i') + \frac{i-1}{2} & \text{if } i \text{ is odd, } 2 \leq i \leq n, 2 \leq r \leq m \\
f(v_r') + \frac{i}{2} & \text{if } i \text{ is even, } 2 \leq i \leq n, 2 \leq r \leq m
\end{cases}
\]

It is easy to see that \( S = \{ f(x) + f(y) : xy \in E(G) \} \)

\[
= \left\{ \frac{n+3}{2}, \frac{n+5}{2}, ..., n+10m-7 \right\}
\]

is a set of \( n(2m-1) \) consecutive integers. Hence, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( G \) with valence

\[
k = p + q + s = \frac{6mn-n+3}{2}.
\]

Example Figure 3.2.15 shows the super edge-magic labeling of the graph \( G \) with \( n = 5 \) and \( m = 4 \).
Figure 3.2.15
3.3 Some general results on super edge-magic graphs

In this section we first study the relation between super edge-magic graphs and strongly indexable graphs.

**Theorem 3.3.1** Every strongly indexable graph is super edge-magic.

**Proof.** Let $f$ be a strong indexer of the $(p, q)$ graph $G$.

Then $\{ f(u) + f(v) : uv \in E(G) \} = \{ 1, 2, 3, \ldots, q \}$.

Now, define $g : V(G) \rightarrow \{ 1, 2, \ldots, p \}$ to be the bijection such that $g(v) = f(v) + 1$ for each vertex $v$ of $G$. Then,

$\{ g(u) + g(v) : uv \in E(G) \} = \{ f(u) + f(v) + 2 : uv \in E(G) \} = \{ 3, 4, 5, \ldots, q + 2 \}$. By Lemma 1.2.2, it follows that $g$ extends to a super edge-magic labeling of $G$ with valence $k = p + q + 3$.

**Theorem 3.3.2** Let $G$ be a super edge-magic graph and $f$ be a super edge magic labeling of $G$. Then $G$ is strongly indexable if the valence of $f$ is $k = p + q + 3$.

**Proof.** Let $f$ be a super edge-magic labeling of $G$ with valence $k = p + q + 3$. Then, by Lemma 1.2.2, we get

$$S = \{ f(u) + f(v) : uv \in E(G) \}$$

$$= \{ k - (p + 1), k - (p + 2), \ldots, k - (p + q) \}$$

$$= \{ 3, 4, 5, \ldots, (q + 2) \}.$$ Define a function $g : V(G) \rightarrow \{ 0, 1, 2, \ldots, (p - 1) \}$ to be the bijection such that $g(v) = f(v) - 1$ for each vertex $v$ of $G$. Then,
Super edge-magic graphs

Since the degree of every vertex of an Eulerian graph is even, we get

\[ \{ g(u) + g(v) : uv \in E(G) \} = \{ f(u) + f(v) - 2 : uv \in E(G) \} = \{ 1, 2, \ldots, q \}. \]

Hence, \( g \) is a strong indexer of \( G \).

Lemma 1.2.5 establishes the relationship between a super edge-magic graph and a felicitous graph.

In view of the above result we state the next theorem.

**Theorem 3.3.3** If \( G \) is a strongly indexable \((p, q)\)-graph with \( q \geq p - 1 \), then \( G \) is felicitous.

**Proof.** Follows from Theorem 3.3.1 and Lemma 1.2.5.

**Theorem 3.3.4** If \( f \) is a super edge-magic labeling of an Eulerian \((p, q)\)-graph \( G \) then, \( q(q + 2s - 1) = 0 \pmod{4} \), where \( s = \min\{ f(x) + f(y) : xy \in E(G) \} \).

**Proof.** Let \( f \) be a super edge-magic labeling of the Eulerian \((p, q)\)-graph \( G \). Then, by Lemma 1.2.2, the set

\[ S = \{ f(x) + f(y) : xy \in E(G) \} \]

consists of \( q \) consecutive integers and valence of \( f \) is \( k = p + q + s \). Hence,

\[ \sum_{v \in V(G)} f(v) d(v) = \sum_{z \in S} z = qk - qp - \frac{q(q + 1)}{2} = q(k - p) - \frac{q(q + 1)}{2} \]

\[ = q(q + s) - \frac{q(q + 1)}{2} = \frac{2q(q + s) - q(q + 1)}{2} \]

\[ = \frac{q(q + 2s - 1)}{2}. \]
Since the degree of every vertex of an eulerian graph is even, we get
\[ \frac{q(q + 2s - 1)}{2} \] is even. Therefore, \( q(q + 2s - 1) \) is a multiple of 4.

**Theorem 3.3.5** If \( G \) is a connected super edge-magic \( (p, q) \)-graph then, \( s \leq p + 1 \) for any super edge-magic labeling \( f \) of \( G \) where \( s = \min\{ f(u) + f(v) : uv \in E(G) \} \).

**Proof.** Let \( G \) be a connected super edge-magic graph and \( f \) be a super edge-magic labeling of \( G \) with \( s = \min\{ f(u) + f(v) : uv \in E(G) \} \). Then, \( s = \{ f(u) + f(v) : uv \in E(G) \} = \{ s, s + 1, s + 2, \ldots , s + q - 1 \} \).

Since \( f : V (G) \rightarrow \{ 1, 2, \ldots , p \} \), \( s + q - 1 \leq 2p - 1 \) and we get \( s + q \leq 2p \). If \( s \geq p + 2 \), then \( p + 2 + q \leq 2p \) so that \( q \leq p - 2 \). This is a contradiction to the connectedness of \( G \). This completes the proof of the theorem.

**Corollary 3.3.1** The upperbound given by Theorem 3.3.5 is sharp.

**Proof.** To see this, consider the star \( K_{1,n} \). Let \( v \) be the central vertex and \( v_1, v_2, \ldots , v_n \) be the end-vertices of \( K_{1,n} \).

Consider the function \( f : V (K_{1,n}) \rightarrow \{ 1, 2, \ldots , n + 1 \} \) such that \( f(v) = n + 1 \), \( f(v_i) = i, \ 1 \leq i \leq n \).
Note that \( S = \{ f(u) + f(v) : uv \in E(K_{1,n}) \} \)

\[
= \{ n + 2, n + 3, \ldots, (2n + 1) \} \text{ is a set of } n \text{ consecutive integers. Therefore, by Lemma 1.2.2, } f \text{ extends to a super edge-magic labeling of } K_{1,n} \text{ and } s = n + 2 = p + 1.
\]

**Theorem 3.3.6** Let \( G \) be a \((p, q)\)-graph and \( f \) be a super edge-magic labeling of \( G \) with valence \( k \). Then,

\[
\frac{1}{2q} [(\Delta-1)p(p+1)+(p+q(p+q+1)) \leq k \leq \frac{1}{2q} [(\Delta-1)p(p+1)+(p+q(p+q+1)]
\]

**Proof.** Since \( k \) is the valence of \( f \), \( f(u) + f(v) + f(uv) = k \) for every edge \( uv \in E(G) \). Now, adding all the constants obtained at each edge of \( G \), we get

\[
qk = \sum_{u \in V(G)} f(u)d(u) + \sum_{u \in E(G)} f(u)
\]

\[
\leq \Delta \sum_{u \in V(G)} f(u) + \sum_{u \in E(G)} f(u)
\]

\[
= (\Delta - 1) \sum_{u \in V(G)} f(u) + \sum_{u \in E(G)} f(u)
\]

\[
= (\Delta - 1) \frac{p(p+1)}{2} + [1 + 2 + 3 + \ldots + (p + q)]
\]

\[
= (\Delta - 1) \frac{p(p+1)}{2} + \frac{(p + q)(p + q + 1)}{2}
\]

\[
= \frac{1}{2} [(\Delta - 1)p(p+1) + (p + q)(p + q + 1)]
\]
Therefore, \( k \leq \frac{1}{2q} [(\Delta - 1)p(p + 1) + (p + q)(p + q + 1)] \). By a similar argument with \( \Delta \) replaced by \( \delta \) and \( \leq \) by \( \geq \) we get

\[
k \geq \frac{1}{2q} [(\delta - 1)p(p + 1) + (p + q)(p + q + 1)]
\]

and the proof is complete.

\[\square\]

**Corollary 3.3.2** If \( f \) is a super edge-magic labeling of an \( r \)-regular \((p, q)\)-graph \( G \) with valence \( k \), then

\[
k = \frac{1}{2q} [(r - 1)p(p + 1) + (p + q)(p + q + 1)]
\]

**Proof.** Apply Theorem 3.3.6 with \( \delta = \Delta = r \).

\[\square\]