Chapter IV

SUPER MAGIC STRENGTH AND SUPER EDGE-MAGIC DEFICIENCY OF GRAPHS

4.1 Introduction

Motivated by the investigations of the super magic strengths of a graph, the super edge-magic deficiencies of graphs were extensively studied by Figueroa-Centeno et al. They proved that

\[ \Delta_{em}(P_n \cup K_{1, n}) = \frac{n}{2} \]

is finite for all finite graphs. When \( n \) is even,

They also proved that

\[ \Delta_{st}(P_n \cup K_{1, n}) = 0 \]
CHAPTER IV

SUPER MAGIC STRENGTH AND SUPER EDGE-MAGIC DEFICIENCY OF GRAPHS

4.1 Introduction

Avadayappan et al. [11] introduced the notion of super magic strength of a graph. They denoted super magic strength of a graph $G$ by $sm(G)$ and obtained the following results:

$$sm(P_{2n}) = 5n + 1; \quad sm(P_{2n+1}) = 5n + 3; \quad sm(C_{2n+1}) = 5n + 4$$ and $$sm(B_{n,n}) = 5n + 6,$$ where $B_{n,n}$ is the $n$-bistar.

Motivated by the definition of edge-magic deficiency of a graph, Figueroa-Centeno et al. [69] defined the concept of super edge-magic deficiency of a graph in 1999. They obtained the following results on the super edge-magic deficiency of graphs: $\mu_s(K_n)$ is infinite if and only if $n \geq 5$; $\mu_s(nK_2) = (n - 1) \pmod{2}$; $\mu_s(K_{m,n}) \leq (m - 1)(n - 1)$ and $\mu_s(F)$ is finite for all forests $F$. Figueroa-Centeno et al. [73] proved that $\mu_s(P_m \cup K_{1,n}) = 1$ if $m = 2$ and $n$ is odd or $m = 3$ and is not congruent to 0 mod 3 while in all other cases $\mu_s(P_m \cup K_{1,n}) = 0$. They also proved that $\mu_s(2K_{1,n}) = 1$ when $n$ is odd and $\mu_s(2K_{1,n}) \leq 1$ when $n$ is even.
We note the following fact also. Let \( f \) be a super edge-magic labeling of a \((p, q)\)-graph \( G \) with valence \( k \). Then, \( f(u) + f(v) + f(uv) = k \) for every edge \( uv \in E(G) \). Now, adding all the constants obtained at each edge of \( G \), we get
\[
qk = \sum_{u \in V(G)} f(u) d(u) + \sum_{uv \in E(G)} f(uv)
\]

(1)

4.2 On the super magic strength of graphs

In this section, we establish the super magic strength of some well known graphs.

**Theorem 4.2.1** \( sm(C_n \odot P_2) = \frac{15n + 3}{2} \) for all odd \( n \geq 3 \).

**Proof.** By Theorem 3.2.5, \( sm(C_n \odot P_2) \leq \frac{15n + 3}{2} \) for all odd \( n \geq 3 \).

Let \( f \) be a super edge-magic labeling of the graph \( C_n \odot P_2 \) with valence \( k \). Then, using (1) with \( p = 3n \) and \( q = 4n \) we get

\[
4nk = 2 \sum_{i=1}^{n} f(a_i) + 2 \sum_{i=1}^{n} f(b_i) + 4 \sum_{i=1}^{n} f(v_i) + \sum_{e \in E} f(e)
\]

where

\[
a_i, b_i \ (1 \leq i \leq n)
\]

are the vertices adjacent to the rim vertices \( v_i \) of \( C_n \) in \( C_n \odot P_2 \). Again,

\[
4nk = \left[ \sum_{i=1}^{n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) + \sum_{e \in E} f(e) \right] + \\
\left[ \sum_{i=1}^{n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) \right] + 2 \sum_{i=1}^{n} f(v_i)
\]

For \( C_n \odot P_2 \), since it has \( 4n \) vertices and on edge
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Let \( n \geq 1 \) and \( i \leq n \). Denote the vertices of degree 3 and \( v_i, 1 \leq i \leq n \) denote the vertices of degree 5 in \( G \). Hence, using (1) and (2) with \( p = 4n \) and \( q = 6n \) we get

\[
6nk = 2 \sum f(a_i) + 3 \sum f(b_i) + \sum f(v_i) = 2n(7n + 1) + 3n(3n + 1) + \sum f(v_i)
\]

Thus, \( k = \frac{7(7n + 1)}{8} + \frac{3(3n + 1)}{8} + \frac{1}{2n} \sum f(v_i) \)

\[
= \frac{58n + 10}{8} + \frac{1}{2n} \frac{n(n + 1)}{2}
\]

\[
= \frac{29n + 5}{4} + \frac{n + 1}{4}
\]

\[
= \frac{30n + 6}{4}
\]

\[
= \frac{15n + 3}{2}
\]

i.e., \( k \geq \frac{15n + 3}{2} \)

Thus, \( sm(C_n \odot P_2) \geq \frac{15n + 3}{2} \) for all odd \( n \geq 3 \).

Therefore, \( sm(C_n \odot P_2) = \frac{15n + 3}{2} \) for all odd \( n \geq 3 \).

**Theorem 4.2.2** \( \frac{20n + 3}{2} \leq sm(C_n \odot P_3) \leq \frac{21n + 3}{2} \) for all odd \( n \geq 3 \).

**Proof.** By Theorem 3.2.6, \( sm(C_n \odot P_3) \leq \frac{21n + 3}{2} \) for all odd \( n \geq 3 \).

The graph \( C_n \odot P_3 \) has 4n vertices and 6n edges.
Let $a_i$ ($1 \leq i \leq 2n$) denote the vertices of degree 2, $b_i$ ($1 \leq i \leq n$) denote the vertices of degree 3 and $v_i$ ($1 \leq i \leq n$) denote the vertices of degree 5 in $C_n \circ P_3$.

Let $f$ be a super edge-magic labeling of the graph $C_n \circ P_3$ with valence $k$. Then, using (1) with $p = 4n$ and $q = 6n$ we get

$$6nk = 2\sum_{i=1}^{2n} f(a_i) + 3\sum_{i=1}^{n} f(b_i) + 5\sum_{i=1}^{n} f(v_i) + \sum_{e \in E} f(e)$$

$$= \left( \sum_{i=1}^{2n} f(a_i) + \sum_{i=1}^{n} f(b_i) + \sum_{i=1}^{n} f(v_i) \right) + \sum_{i=1}^{2n} f(a_i) + 3\sum_{i=1}^{n} f(v_i)$$

Thus, $k = \frac{1}{6n} \left( 5n(10n + 1) + 2n(4n + 1) + 1 + 2 + \cdots + n \right)$

$$= \frac{1}{6n} \left[ 5n(10n + 1) + 2n(4n + 1) + \frac{n(n + 1)}{2} + 3 \frac{n(n + 1)}{2} \right]$$

$$= \frac{1}{6} \left[ \frac{58n + 7 + n + 4}{2} + \frac{3(n + 1)}{2} \right]$$

$$= \frac{1}{6} \left[ \frac{116n + 14 + 4n + 4}{2} \right]$$

$$= \frac{20n + 3}{2}$$
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Hence, \( sm(C_n \odot P_3) \geq \frac{20n + 3}{2} \) for all odd \( n \geq 3 \).

Therefore, \( \frac{20n + 3}{2} \leq sm(C_n \odot P_3) \leq \frac{21n + 3}{2} \) for all odd \( n \geq 3 \).

\[ \square \]

**Theorem 4.2.3** Let \( G \) denote the graph consisting of an odd cycle \( C_n (n \geq 7) \) with a chord joining two vertices at a distance 3. Then,

\[ sm(G) = \frac{5n + 3}{2} \] for all odd \( n \geq 7 \).

**Proof.** Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Next join the vertices \( v_1 \) and \( v_{n-2} \) as a chord for \( G \) so that \( d(v_1, v_{n-2}) = 3 \). Note that \( G \) has \( n \) vertices and \( n+1 \) edges. By Theorem 3.2.10, \( sm(G) \leq \frac{5n + 3}{2} \) for all odd \( n \geq 7 \).

Let \( f \) be a super edge-magic labeling of the graph \( G \) with valence \( k \).

Then, using (1) with \( p = n \) and \( q = n + 1 \) we get

\[
(n + 1)k = 2 \sum_{i=1}^{n} f(v_i) + 3f(v_i) + 3f(v_{n-2}) + \sum_{e \in E(G)} f(e)
\]

\[
= \left[ \sum_{i=1}^{n} f(v_i) \right] + \frac{n(n + 1)}{2} + f(v_1) + f(v_{n-2})
\]

\[
= \frac{(2n + 1)(2n + 2)}{2} + \frac{n(n + 1)}{2} + f(v_1) + f(v_{n-2})
\]

Hence, \( k = \frac{(2n + 1)(n + 1)}{(n + 1)} + \frac{n(n + 1)}{2(n + 1)} + \frac{f(v_1) + f(v_{n-2})}{(n + 1)} \)

\[
\geq (2n + 1) + \frac{n}{2} + \frac{3}{(n + 1)}
\]
greater than \((2n + 1) + \frac{n}{2}\)

\[\frac{5n + 2}{2}\]

Since \(n\) is odd, \(k > \frac{5n + 2}{2}\) implies \(k \geq \frac{5n + 3}{2}\).

Thus \(sm(G) \geq \frac{5n + 3}{2}\) for all odd \(n \geq 7\).

Therefore, \(sm(G) = \frac{5n + 3}{2}\) for all odd \(n \geq 7\).

\[\Box\]

### 4.3 Super edge-magic deficiencies of graphs

In this section we study the super edge-magic deficiencies of some graphs. We have proved (Theorem 3.2.9) that the graph \(\overline{K_n} + 2K_2\) is not super edge-magic. Now, we prove the following

**Theorem 4.3.1** If \(n\) is odd then, \(\mu_s(\overline{K_n} + 2K_2) = +\infty\).

**Proof.** The order and size of the graph \(\overline{K_n} + 2K_2\) are given by \(p = n + 4\) and \(q = 4n + 2\) so that \(q \equiv 2 \pmod{4}\). If \(n\) is odd then, the degree of every vertex of \(\overline{K_n} + 2K_2\) is even. By Theorem 1.2.1, it follows that \(\mu_s(\overline{K_n} + 2K_2) = +\infty\) if \(n\) is odd.

**Theorem 4.3.2** If \(n\) is odd then, \(\mu_s(K_2 \odot C_n) \leq \frac{n-3}{2}\).

**Proof.** The graph \(K_2 \odot C_n\) is obtained by attaching a cycle \(C_n\) of \(n\) vertices to the two end-vertices of \(K_2\). Let \(u_1, u_2\) be the two vertices
of $K_2$ and $v_1, v_2, \ldots, v_n$ be the $n$ vertices of the cycle $C_n$ which is attached with $u_1$ and $w_1, w_2, \ldots, w_n$ be the $n$ vertices of the cycle $C_n$ which is attached with $u_2$. Note that $K_2 \odot C_n$ has $2n + 2$ vertices and $4n + 1$ edges. Let $a_1, a_2, \ldots, a_t$ denote the isolated vertices, where $t = \frac{n - 3}{2}$. Let $G = (K_2 \odot C_n) \cup tK_1$.

Consider the vertex labeling $f : V(G) \rightarrow \left\{1, 2, 3, \ldots, \frac{5n + 1}{2}\right\}$ such that

$$f(x) = \begin{cases} \frac{i + 1}{2} & \text{if } x = v_i, \ i \text{ is odd, } 1 \leq i \leq n \\ \frac{n + i + 1}{2} & \text{if } x = v_i, \ i \text{ is even, } 1 \leq i \leq n \\ \frac{3n + i + 2}{2} & \text{if } x = w_i, \ i \text{ is odd, } 1 \leq i \leq n \\ (2n + 1) + \frac{i}{2} & \text{if } x = w_i, \ i \text{ is even, } 1 \leq i \leq n \\ n + 1 & \text{if } x = a_i, \ 1 \leq i \leq \frac{n - 3}{2} \\ n + i + 1 & \text{if } x = a_i, \ 1 \leq i \leq \frac{n - 3}{2} \end{cases}$$

Note that $S = \{f(x) + f(y) : xy \in E(G)\}$

$$= \left\{\left(\frac{n + 1}{2}\right) + 1, \left(\frac{n + 1}{2}\right) + 2, \ldots, \left(\frac{n + 1}{2}\right) + (4n + 1)\right\}$$

is a set of
$4n + 1$ consecutive integers. Hence, by Lemma 1.2.2, $f$ extends to a super edge-magic labeling of $G$. Therefore, $\mu_s(K_2 \odot C_n) \leq \frac{n - 3}{2}$ if $n$ is odd, $n \neq 3$.

The super edge-magic labeling of $K_2 \odot C_3$ is given in Figure 4.3.1.

Hence, $\mu_s(K_2 \odot C_3) = 0$. This completes the proof of the theorem. \qed
Example Figure 4.3.2 gives the super edge-magic labeling of \((K_2 \odot C_t) \cup 2K_1\).

Define a vertex labeling \(f: V(G) \rightarrow \{1, 2, ..., 3n\}\) such that:

- \(f(v_0) = 1\)
- \(f(v_i) = 3t - 2i + 3t - 1\) for \(1 \leq i \leq n\)
- \(f(v_i) = 2(n + 1) - i\) for \(1 \leq i \leq n - 1\)

Figure 4.3.2

Theorem 4.3.3 \(\mu_s(W_n \odot K_1) \leq (n - 1)\) for all odd \(n \geq 3\).

Proof. Let \(W_n\) be a wheel on \(n + 1\) vertices. Attach a pendant edge to each rim-vertex of \(W_n\). The graph so obtained is denoted by \(W_n \odot K_1\).

Note that \(W_n \odot K_1\) consists of \(2n + 1\) vertices and \(3n\) edges.

Let \(v_1, v_2, ..., v_n\), be the rim vertices of \(W_n\) and \(v_0\) be the central vertex of \(W_n\). Let \(u_1, u_2, ..., u_n\), be the end-vertices adjacent
to the corresponding $v_i$, $s$, $i = 1, 2, \ldots, n$. Let $a_1, a_2, \ldots, a_t$ be the isolated vertices, where $t = n - 1$. Let $G = (W_n \odot K_1) \cup t K_1$.

Define a vertex labeling $f : V(G) \to \{1, 2, \ldots, 3n\}$ such that

$$f(v_0) = 1$$

$$f(v_{2i-1}) = 3i - 1 \quad \text{for} \quad 1 \leq i \leq \frac{n+1}{2}$$

$$f(v_{2i}) = 3i \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f(u_i) = 3n - 3i + 4 \quad \text{for} \quad 2 \leq i \leq n$$

$$f(u_1) = 3n$$

$$f(a_{2i-1}) = \frac{3(n+1)}{2} + 3(i-1) \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f(a_{2i}) = \frac{3(n+1)}{2} + 2 + 3(i-1) \quad \text{for} \quad 2 \leq i \leq n-1$$

Note that $S = \{f(x) + f(y) : xy \in E(G)\} = \{3, 4, 5, \ldots, 3n+2\}$ consists of $3n$ consecutive integers. Hence, by Lemma 1.2.2, $f$ extends to a super edge-magic labeling of $G$.

Therefore, $\mu_s(W_n \odot K_1) \leq (n - 1)$ for all odd $n \geq 3$.

\textbf{Example} \quad Figure 4.3.3 shows the super edge-magic labeling of $(W_6 \odot K_1) \cup 8K_1$. 

\hspace{1cm} □
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Theorem 4.3.4 Let $G$ be a super edge-magic $(p, q)$-graph and $f$ be a super edge-magic labeling of $G$. Then,

$$\mu_s(G + \overline{K_n}) \leq s + q + n(p - 1) - 2p - 1$$

for all $n \geq 1$, where

$$s = \min \{ f(x) + f(y) : xy \in E(G) \}$$
Proof. Let \( v_1, v_2, \ldots, v_p \) be the vertices of \( G \). Since \( f \) is a super edge-magic labeling of \( G \), \( S = \{ f(x) + f(y) : xy \in E(G) \} = \{ s, s+1, s+2, \ldots, s+q-1 \} \) where \( s = \min \{ f(x) + f(y) : xy \in E(G) \} \).

Let \( V(K_n) = \{ y_1, y_2, \ldots, y_n \} \). Note that the graph \( G + \overline{K_n} \) has \( p+n \) vertices and \( q+np \) edges.

Let \( t = s + q - 1 - (p+1) + (n-1)(p-1) = t_1 + (n-1)(p-1) \)
where \( t_1 = s + q - 1 - (p+1) \). Also \( t = s + q + n(p-1) - 2p - 1 \).

Let \( w_r \) and \( z_{ij} \), \( 1 \leq r \leq t_1, 1 \leq i \leq (n-1), 1 \leq j \leq (p-1) \), be the \( t \) isolated vertices. The graph \( H = (G + \overline{K_n}) \cup tK_1 \) has \( p + n + t \) vertices and \( q + np \) edges.

Define a vertex labeling \( g : V(H) \to \{ 1, 2, \ldots, p + n + t \} \) such that

\[
\begin{align*}
g(x) &= \begin{cases} 
  f(x) & \text{if } x \in V(G) \\
  s + q - 1 + p(i-1) & \text{if } x = y_i, 1 \leq i \leq n \\
  p + i & \text{if } x = w_i, 1 \leq i \leq (s + q - p - 2) = t_1 \\
  s + q + (i-1)p + j - 1 & \text{if } x = z_{ij}, 1 \leq i \leq (n-1), 1 \leq j \leq (p-1) 
\end{cases}
\end{align*}
\]

Note that \( S = \{ g(x) + g(y) : xy \in E(H) \} \)
\[
= \{ s, s+1, s+2, \ldots, s+q-1, \ldots, s+q-1+np \}
\]
is a set of \( q+np \) consecutive integers. Hence, by Lemma 1.2.2, \( g \) extends to a super edge-magic labeling of \( H \). Therefore, \( \mu_s(G + \overline{K_n}) \leq t \) for all \( n \geq 1 \).

\[ \text{i.e., } \mu_s(G + \overline{K_n}) \leq s + q + n(p-1) - 2p - 1 \text{ for all } n \geq 1. \]

Example Figure 4.3.4 shows the super edge-magic labeling of \( C_5 \odot K_1 \). Figure 4.3.5 shows the vertex labeling of \( (C_5 \odot K_1) + \overline{K_2} \cup 11K_1 \).
Theorem 4.3.5. $1 \leq \mu_s(P_n \times K_4) = \delta(P_n \times K_4)$ for all $n \geq 1$.

Proof. The graph $P_n \times K_4$ has $4n$ vertices and $10n - 4$ edges. Since $\delta(P_n \times K_4) \geq 3$, by Corollary 4.2.1, $P_n \times K_4$ is not super edge-magic.

Hence $\mu_s(P_n \times K_4) \geq 1$. Let $\gamma$ be any edge of $G$, $G = P_n \times K_4$. Let $\nu_1, \nu_2, \nu_3, \nu_4 : 1 \leq i \leq 4$ be the vertex set of $G$. Denote the isolated vertices by $a_1, a_2, \ldots, a_n$. Let $G = G_1 \cup \ldots \cup G_n$. Consider the vertex labeling $f : V(G) \rightarrow \{1, 2, \ldots, 5n\}$ such that

$$f(v_{2i-1}) = 10(i-1), \quad 1 \leq i \leq \frac{n}{2} \quad \text{if } n \text{ is even}$$

$$f(v_{2i-1}) = 10(i-1), \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{if } n \text{ is odd}$$

$$f(v_{2i}) = 10i, \quad 1 \leq i \leq \frac{n}{2} \quad \text{if } n \text{ is even}$$

$$f(v_{2i}) = 10i, \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{if } n \text{ is odd}$$

Figure 4.3.4

Figure 4.3.5
Theorem 4.3.5 \[ 1 \leq \mu_s( P_n \times K_4 ) \leq n \] for all \( n \geq 1 \).

Proof. The graph \( P_n \times K_4 \) has \( 4n \) vertices and \( 10n - 4 \) edges. Since \( \delta( P_n \times K_4 ) > 3 \), by Corollary 1.2.1, \( P_n \times K_4 \) is not super edge-magic.

Hence \( \mu_s( P_n \times K_4 ) \geq 1 \) for all \( n \geq 1 \). Let \( G_1 = P_n \times K_4 \). Let \( \{ v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4} : 1 \leq i \leq n \} \) be the vertex set of \( G_1 \). Denote the isolated vertices by \( a_1, a_2, \ldots, a_n \). Let \( G = G_1 \cup nK_1 \). Consider the vertex labeling \( f : V(G) \rightarrow \{ 1, 2, \ldots, 5n \} \) such that

\[
f(v_{2i-1,1}) = 10i - 5, \quad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}
\]

\[
1 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd}
\]

\[
f(v_{2i-1,2}) = 1 + 10(i - 1), \quad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}
\]

\[
1 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd}
\]

\[
f(v_{2i-1,3}) = 10i - 7, \quad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}
\]

\[
1 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd}
\]

\[
f(v_{2i-1,4}) = 10i - 8, \quad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}
\]

\[
1 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd}
\]

\[
f(v_{2i,1}) = 10i - 3, \quad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}
\]

\[
1 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd}
\]
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\[ f(v_{2i, 2}) = 10 \cdot i, \quad 1 \leq i \leq \frac{n}{2} \quad \text{if } n \text{ is even} \]

\[ 1 \leq i \leq \frac{n-1}{2} \quad \text{if } n \text{ is odd} \]

\[ f(v_{2i, 3}) = 10 \cdot i - 4, \quad 1 \leq i \leq \frac{n}{2} \quad \text{if } n \text{ is even} \]

\[ 1 \leq i \leq \frac{n-1}{2} \quad \text{if } n \text{ is odd} \]

\[ f(v_{2i, 4}) = 10 \cdot i - 2, \quad 1 \leq i \leq \frac{n}{2} \quad \text{if } n \text{ is even} \]

\[ 1 \leq i \leq \frac{n-1}{2} \quad \text{if } n \text{ is odd} \]

\[ f(a_i) = 4 + 5 \cdot (i - 1), \quad 1 \leq i \leq n \]

Then it is easy to see that

\[ S = \{ f(x) + f(y) : xy \in E(G) \} = \{ 3, 4, 5, \ldots, (10n - 2) \} \]

is a set of \(10n - 4\) consecutive integers. Hence, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \(G\).

Therefore, \( \mu_s(P_n \times K_4) \leq n \) for all \( n \geq 1 \). This completes the proof of the theorem. 

Example Figure 4.3.6 shows the super edge-magic labeling of \(P_n \times K_4 \cup 4K_1\).
Theorem 4.3.6 \[ \mu_s(P_n[P_2]) \leq \left\lceil \frac{n-1}{2} \right\rceil \] for all \( n > 1 \).

Proof. Let \( G = P_n \cup K_1 \) be the vertices of the path \( P_n \) and \( v_1, v_2 \) be the vertices of \( P_2 \). Let \( G = \{ 1, 2, \ldots, 2n + \left\lceil \frac{n-1}{2} \right\rceil \} \) such that \( f(\{u_i, u_{i+1}\}) = 5i - 2 \), for \( 1 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil \). Note that \( S = \{ f(xy) \mid xy \in E(G) \} \setminus \{3, 4, 5, \ldots, 5n-2\} \).

Therefore, \( \mu_s(P_n[P_2]) \leq \left\lceil \frac{n-1}{2} \right\rceil \) for all \( n > 1 \).

Figure 4.3.6
Theorem 4.3.6 \( 1 \leq \mu_s(P_n[P_2]) \leq \left\lfloor \frac{n-1}{2} \right\rfloor \) for all \( n > 1 \).

Proof. The graph \( P_n[P_2] \) has \( 2n \) vertices and \( 5n - 4 \) edges.

Let \( u_1, u_2, \ldots, u_n \) be the vertices of the path \( P_n \) and \( v_1, v_2 \) be the vertices of \( P_2 \). Let \( a_1, a_2, \ldots, a_t \) be the isolated vertices, where \( t = \left\lfloor \frac{n-1}{2} \right\rfloor \). Let \( G = P_n[P_2] \cup tK_1 \).

Consider the vertex labeling \( f: V(G) \rightarrow \{1, 2, \ldots, 2n + \left\lfloor \frac{n-1}{2} \right\rfloor \} \) such that

\[
\begin{align*}
&f\{ (u_{2i+1}, v_1) \} = 5i + 1, \quad 0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \\
f\{ (u_{2i}, v_1) \} = 5i - 2, \quad 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \\
f\{ (u_{2i+1}, v_2) \} = 5i + 2, \quad 0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \\
f\{ (u_{2i}, v_2) \} = 5i, \quad 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \\
f(a_i) = 5i - 1, \quad 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor 
\end{align*}
\]

Note that \( S = \{ f(x) + f(y) : xy \in E(G) \} = \{ 3, 4, 5, \ldots, (5n - 2) \} \) is a set of \((5n - 4)\) consecutive integers. Hence, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( G \).

Therefore, \( \mu_s(P_n[P_2]) \leq t \) for all \( n > 1 \).

i.e., \( \mu_s(P_n[P_2]) \leq \left\lfloor \frac{n-1}{2} \right\rfloor \) for all \( n > 1 \).
For the graph $P_n[P_2]$, $q = 5n - 4$ and $2p - 3 = 4n - 3$ so that $q > 2p - 3$ for $n > 1$. So by Lemma 1.2.1, $P_n[P_2]$ is not super edge-magic for $n > 1$. Hence, $\mu_s(P_n[P_2]) \geq 1$ and we get $1 \leq \mu_s(P_n[P_2]) \leq \left\lceil \frac{n - 1}{2} \right\rceil$ for all $n > 1$.

Example Figure 4.3.7 gives the super edge-magic labeling of $P_s[P_2] \cup 4K_1$.

Let $K_{m,n} + e$ denote the graph obtained when we add an edge $e$ to the graph $K_{m,n}$, where $m$ or $n \geq 2$. 

![Figure 4.3.7](image-url)
Theorem 4.3.7 \[ \mu_s(K_{m,n} + e) \leq (m-2)(n-1) \] for \( m \) or \( n \geq 2 \).

**Proof.** Let \( V(K_{m,n}) = \{ u_1, u_2, \ldots, u_m; v_1, v_2, \ldots, v_n \} \). Without loss of generality, let \( m \geq 2 \) and the edge \( e = u_1u_2 \). Let \( a_{ij}, 1 \leq i \leq (m-2), 1 \leq j \leq (n-1) \) be the isolated vertices.

Let \( G = (K_{m,n} + e) \cup (m-2)(n-1)K_1 \). Consider the vertex labeling \( f : V(G) \rightarrow \{ 1, 2, \ldots, mn + (m-2)(n-1) \} \) such that

\[ f(u_1) = 1, \quad f(u_i) = (i-1)n + 2, \quad 2 \leq i \leq m \]

\[ f(v_i) = i+1, \quad 1 \leq i \leq n, \quad f(a_{ij}) = in + 2 + j, \quad 1 \leq i \leq (m-2), \quad 1 \leq j \leq (n-1). \]

It is easy to see that \( S = \{ f(x) + f(y) : xy \in E(G) \} \)

\[ = \{ 3, 4, 5, \ldots, mn + 3 \} \]

Hence, by Lemma 1.2.2, \( f \) extends to a super edge-magic labeling of \( G \).

Therefore, \( \mu_s(K_{m,n} + e) \leq (m-2)(n-1) \) for \( m \) or \( n \geq 2 \).

**Example** Figure 4.3.8 shows the vertex labeling of \( (K_{4,5} + e) \cup 8K_1 \) defined above.

![Figure 4.3.8](image-url)