Chapter V

RELIABILITY CONCEPTS IN DISCRETE TIME

5.1 Preliminaries

The focal theme of the last two chapters has been the characterization of discrete and continuous distributions which share the common property that their mean residual life is of linear form. Apart from this property that has immense value to reliability modelling, most other features relating to these models have also some significant implications in the context of reliability analysis. In the present chapter, we look more closely at some concepts that are frequently used in life length studies vis a vis their relationships, with reference to various notions discussed earlier along with their inherent properties.

A failure time distribution represents an attempt to describe mathematically the length of life of a component or device. However, our inability to isolate the vast body of causes, that individually or collectively are responsible for the failure of the device at a particular instant, often renders, the identification of the failure distribution very difficult. Available at the disposal of the analyst is only some actual observations on the time to failure and these have to be made use of to explore a plausible model. When
the data set indicates the desirability of a skew distribution, the problem becomes even more difficult, as asymmetric models differ markedly at the tails and the actual observations at the right tail are sparse on account of limited sample size. This has led to several concepts that enable differentiation between various models based upon physical considerations that governed the failure phenomenon. Some of these, such as failure rate, mean (median) residual life function, equilibrium distribution along with the essential conditions under which their properties provide specific models have already been discussed. Yet another and perhaps more versatile way of describing the failure mechanism is to expose the manner in which its life length is affected by the advancement of age. In other words, one can check whether the life length of the device is increasing, decreasing or remaining steady together with an assessment of the manner in which these improvements or deterioration in the effectiveness of the device takes place with regard to its age. The various concepts designed for this purpose are called criteria for ageing. The vast majority of literature on the various criteria for ageing treats life time as continuous with only occasional references to the discrete. Recently there is some spurt of activity towards reliability analysis in the discrete time domain.
Xekalaki (1983a), points out that limitations of measuring devices and the fact that discrete models provide good approximations to their continuous counterparts, necessitate assessment of reliability in discrete time. Further discrete models do occur in a natural way as in fatigue studies the time to failure is measured in terms of the number of cycles to failure which is obviously integer valued. Accordingly elaboration of various concepts analogous to those in the continuous cases become necessary to distinguish classes of life distributions based on the notions of ageing. The definitions of the various classes, their characterizations and some implications among them are discussed in the following sections. It may be noticed that Klefsjö (1982) have touched upon the definitions of the various classes introduced below.

5.2 Increasing (Decreasing) Failure Rate

We first present the oldest and perhaps the simplest concept of ageing based on the monotone character of the failure rate.

**Definition 5.1.**

A discrete random variable $X$ or the corresponding survival function $R(x) = P[X > x]$ belongs to the increasing
failure rate or IFR (decreasing failure rate or DFR) class if

\[ h(x) = \frac{P[X=x]}{R(x)} \]

is an increasing (decreasing) function of \( x \), for all \( x \) in \( I \), where \( I \) is the set of non-negative integers.

From the point of view of elucidating the above definition and also in proving some other results, two characterizations of the IFR class will be established.

Throughout the sequel, the proof will be limited to the IFR class, it being understood that by reversing the monotonicity, results for the dual DFR class follow at once.

5.2.1 Characterizations.

Theorem 5.1.

\( X \) is IFR (DFR) if and only if \( R(x+y)/R(x) \) is a decreasing (increasing) function of \( x \) for all \( y \) in \( I \).

**Proof:**

When \( R(x+y)/R(x) \) is an increasing function of \( x \) for all \( y \) we can write,

\[ \frac{R(x+1)}{R(x+y+1)} - \frac{R(x)}{R(x+y)} > 0, \text{ for all } y \text{ in } I. \]
Hence,

$$\frac{R(x+1)}{R(x)} > \frac{R(x+y+1)}{R(x+y)} ,$$

and

$$1 - \frac{R(x+1)}{R(x)} < 1 - \frac{R(x+y+1)}{R(x+y)} ,$$

since, for all \(x\) the ratio \(R(x+1)/R(x)\) does not exceed unity. It now follows that \(h(x) < h(x+y)\) and therefore \(h(x)\) is an increasing function of \(x\). Thus \(X\) is IFR.

The converse is obtained by retracing the above proof from end to the beginning.

Theorem 5.2.

\(X\) is IFR (DFR) if and only if \(H(x,y)\) is an increasing (decreasing) function of \(x\) for all \(y\) in \(I\), where \(H(x,y)\) is the cumulated failure rate in the interval \([x, x+y-1]\) defined by,

$$H(x,y) = \sum_{t=x}^{x+y-1} h(t), \quad x > 0. \tag{5.1}$$

Proof:

First we suppose that \(X\) is IFR. Then for all \(y\) in \(I\),
\[ H(x+1, y) - H(x, y) = \frac{1}{x+1} \sum_{t=y}^{x+y} h(t) - \frac{1}{x} \sum_{t=y}^{x+y-1} h(t), \quad (5.2) \]

\[ x [h(x+y-1) - h(x+y)] > h(x+y), \]

and

\[ h(x+y+1) > \frac{x+y-1}{y} h(x+y), \text{ for all } x > 0 \]

and \( y \) in \( I \) which implies

\[ h(x+y+1) > h(x+y). \]

Since \( h(x) \) is increasing in \( x \),

\[ \frac{1}{x} \sum_{t=y}^{x+y-1} h(t) < h(x+y-1), \]

and therefore,

\[ H(x+1, y) - H(x, y) > \left( \frac{1}{x+1} \right) [h(x+y) - h(x+y-1)], \]

is replaced by another new unit which acts independently of the first. When the renewal of the system is continued indefinitely, Feller (1968) has shown that the asymptotic function \( H(x+1, y) - H(x, y) \) is equal to the distribution function of the residual life \( Y \) of the unit under observation at time \( t \) has the distribution \((4.12)\).

Thus \( H(x, y) \) is increasing with \( x \). Conversely, relation \((5.2)\) gives,

\[ x+y \sum_{t=x}^{x+y-1} h(t) > (x+1) \sum_{t=x}^{x+y-1} h(t), \]

or

\[ xh(x+y) > \sum_{t=x}^{x+y-1} h(t). \]

Deshpande and Pillai (1986) made a comparison between \( R(x) \) and \( G(x) \), the survival functions of \( X \) and \( Y \) in the continuous case as meaningful. Their point of view is that the distribution of the residual life is more rapidly will come off worse in such a comparison. In this sense, the following result is meaningful.
Hence,

\[ x[h(x+y-1) - h(x+y)] \geq h(x+y), \]

and

\[ h(x+y+1) \geq \frac{x+1}{x} h(x+y), \text{ for all } x > 0 \]

and \( y \) in \( I \) which implies

\[ h(x+y+1) \geq h(x+y), \]

and the required conclusions.

In section 4.4 properties were discussed of the distribution based on the partial sums of a discrete model. If \( X \) denotes the life of a component with survival function \( R(x) \), and whenever the component fails it is replaced by another new unit which acts independently of the first. When the renewal of the system is continued indefinitely, Feller (1968) has shown that the asymptotic distribution of the residual life \( Y \) of the unit under observation at time \( t \) has the distribution (4.12).

Deshpande et. al. (1986) considers the comparison between \( R(x) \) and \( G(x) \), the survival functions of \( X \) and \( Y \) in the continuous case as meaningful. Their point of view is that the life distribution of a unit which ages more rapidly will come off worse in such a comparison. In this sense, the following result is meaningful.
Theorem 5.3. If $X$ is IFR then $h(x) \leq k(x)$, for all $x$ in $I$.

(See section 4.4.1 for the definitions of $h(.)$ and $k(.)$.)

Proof:

When $X$ is IFR, from Gupta (1979), $Y$ is also IFR.

This means that

$$k(x+1) \geq k(x),$$

or

$$1 - \frac{k(x+1)}{k(x)} \leq 0.$$

Therefore,

$$k(x+1) + 1 - \frac{k(x+1)}{k(x)} \leq k(x+1),$$

and

$$1 + k(x+1) \left[1 - \frac{1}{k(x)} \right] \leq k(x+1).$$

The left hand side is $h(x+1)$ from equation (4.17) and our result is proved.

5.3 Increasing (Decreasing) Failure Rate Average

The class of distributions distinguished by increasing failure rate average or IFRA (decreasing failure rate average or DFRA) property was introduced.
for continuous random variables by Birnbaum et al. (1966) in an attempt to find a new class of life distributions that reflect the phenomenon of wear-out. They have shown that this class (1) limiting case of no wear; that is, all exponential distributions, (2) preserves the wearing out phenomenon for a system in which the components also have the same behaviour, (3) is the smallest one with properties (1) and (2). Klefsjö (1982) has considered the discrete IFRA class, preferring to define it in terms of the behaviour of $[F(x)]^{1/x}$ where $F(x) = R(x + 1) = P[X > x]$, as in the continuous case. While in the continuous case

$$\frac{1}{x} \log F(x) = \frac{1}{x} \int_{0}^{x} h(t) dt$$

provides the average failure rate in $[0, x]$ no such meaning can be given to $\log F(x)$ in the discrete case. In order to retain the notion of averaging the failure rate, we adopt the following definition of the IFRA class.

**Definition 5.2.**

A discrete random variable $X$ or its survival function belongs to the IFRA class if

$$\frac{1}{x+1} \sum_{t=0}^{x} h(t) \geq \frac{1}{x} \sum_{t=0}^{x-1} h(t),$$

Proof: If
or equivalently, $H(x,o)$ is an increasing function of $x$, for every $x$ in $I$. The DFRA class is defined by reversing the above inequality.

5.3.1 Properties

Directly from the discussions in the previous section we conclude

Theorem 5.4.

1. The IFRA (DRFA) class contains all IFR(DFR) distributions.

2. If $R(x+y)/R(x)$ is a decreasing (increasing) function of $x$, for all $y > 0$, then $X$ is IFRA (DFRA).

3. The function $H(x,y)$ is increasing (decreasing) in $x$ for all $y$ implies that $X$ is IFRA (DFRA).

Theorem 5.5.

If $X$ is IFRA then

$$R(x) R(y) \succ R(x+y),$$

for all $y \succ 0$.

Proof:

By definition 5.2, $X$ belongs to the IFRA class if
\[
\frac{1}{x+1} \sum_{t=0}^{x} h(t) - \frac{1}{x} \sum_{t=0}^{x-1} h(t) > 0,
\]

or
\[
\frac{1}{x+1} \sum_{t=0}^{x} \left[ 1 - \frac{R(t+1)}{R(t)} \right] - \frac{1}{x} \sum_{t=0}^{x-1} \left[ 1 - \frac{R(t+1)}{R(t)} \right] > 0,
\]

or
\[
\frac{1}{x} \sum_{t=0}^{x-1} h(t) < h(0),
\]

which contradicts equation (5.3) and therefore the theorem is proved.

Now, suppose that the condition \( R(x) R(y) > R(x+y) \) is violated for the IFRA class for at least one \( y \), say, \( y=1 \).

Then one should have
\[
\frac{x-1}{x} \sum_{t=0}^{x-1} \frac{R(t+1)}{R(t)} > \frac{x}{x+1} \sum_{t=0}^{x} \frac{R(t+1)}{R(t)}.
\]

This implies
\[
\frac{1}{x} \sum_{t=0}^{x-1} \frac{R(t+1)}{R(t)} > \frac{R(1)}{R(0)},
\]

and
\[
1 - \frac{1}{x} \sum_{t=0}^{x-1} \frac{R(t+1)}{R(t)} < 1 - \frac{R(1)}{R(0)}.
\]
Since,

\[
\frac{R(t+1)}{r(t)} = 1 - h(t),
\]

this should mean that

\[
\frac{1}{x} \sum_{t=0}^{x-1} h(t) < h(0),
\]

which contradicts our hypothesis and therefore the theorem is proved.

**Theorem 5.6.**

If \( X \) belongs to the IFRA class, then \( r(x) \leq r(0) \), for all \( x > 0 \).

**Proof:**

The IFRA nature of \( X \) means that by

**Theorem 5.5,**

\( R(x) R(y) \geq R(x+y) \), for all \( x, y > 0 \).

Then,

\[
R(x+1) \sum_{y=1}^{\infty} R(y) \geq \sum_{y=1}^{\infty} R(x+y+1),
\]

for all \( x > 0 \).
That is,

\[ R(x+1) r(o) R(1) \geq \sum_{y=1}^{\infty} R(t), \]

or

\[ r(o) R(1) \geq \frac{1}{R(x+1)} \sum_{t=x+1}^{\infty} R(t), \]

this gives

\[ r(o) \geq r(x). \]

Theorem 5.7.

The next property concerns a characterization of the IFRA nature of the equilibrium distribution.

Theorem 5.7.

\[ Y \text{ is IFRA if and only if } \frac{1}{x+1} \sum_{o}^{x} \frac{1}{r(t)} \text{ is an increasing function of } x, \text{ for all } x > 0. \]

Proof:

The proof follows from definition 5.2 and equation (4.13).

5.4. Decreasing (Increasing) Mean Residual Life

A chronological review of the development of reliability concepts reveals that the notion of failure
rate was pursued to the more fundamental than the mean residual life. The potential of the mean residual life in describing various laws of failure had received only limited attentions from earlier researchers. In fact, the independence–dependence relationship between \( h(x) \), \( R(x) \) and \( r(x) \) can be reversed and it is possible to look at the failure patterns based on the behaviour of MRL.

At times, the MRL appears to be a better concept than the failure rate once we closely examine the definitions of the two. The failure rate take into account, the behaviour of the survival probabilities at times \( x \) and \( x+1 \), while \( r(x) \) utilizes the entire information about the survival probabilities from age \( x \) onwards till failure. Thus, if an equipment does not fail in the near future, its failure rate may be zero and at the same time, the MRL may be decreasing as failure is bound to occur once the period for which \( h(x) = 0 \) is surpassed. This difference in the behaviour of the two functions has resulted in ageing concepts based on MRL. We first introduce the decreasing mean residual life or DMRL (increasing mean residual life or IMRL) class.

**Definition 5.3.**

A discrete random variable \( X \) or its distribution belongs to the DMRL class if \( r(x) \geq r(x+1) \) and belongs to the dual IMRL class if \( r(x) \leq r(x+1) \) for every \( x \) in \( I \).
5.4.1. Properties.

Theorem 5.7.

A sufficient condition for $X$ to have DMRL(IMRL) is that $R(x+y)/R(x)$ is a decreasing (increasing) function of $x$, for every $y$ in $I$.

Proof:

By hypothesis, $R(x+y+1)/R(x+1)$ is a decreasing function of $x$. Summation over $y$ yields,

$$\frac{1}{R(x+1)} \sum_{y=0}^{\infty} R(x+y+1) \quad \text{or} \quad \frac{1}{R(x+1)} \sum_{y=0}^{\infty} R(y)$$

It is clear that $r(x)$ steadily decreases with $x$ and $h(x)$ is decreasing in $x$. This is the same as saying that $R(x)$ decreases with $x$ for all $y$.

Corollary 5.1.

If $X$ is IFR(DFR), then $X$ is DMRL (IMRL).

Corollary 5.2.

If $H(x+1,y) \geq H(x,y)$, then $X$ is DMRL.

Notice that these corollaries are direct consequences of Theorems 5.1 and 5.2.

It remains to demonstrate that as in the continuous case, DMRL does not imply IFR; otherwise we would not be
introducing a new notion of ageing since the IFR class implies the DMRL class. For the purpose we present the following example.

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.21</td>
<td>0.15</td>
<td>0.22</td>
<td>0.22</td>
<td>0.20</td>
</tr>
<tr>
<td>R(x)</td>
<td>1.00</td>
<td>0.79</td>
<td>0.64</td>
<td>0.42</td>
<td>0.20</td>
</tr>
<tr>
<td>h(x)</td>
<td>0.21</td>
<td>0.19</td>
<td>0.34</td>
<td>0.52</td>
<td>1.00</td>
</tr>
<tr>
<td>r(x)</td>
<td>2.60</td>
<td>2.00</td>
<td>1.50</td>
<td>1.00</td>
<td>0</td>
</tr>
</tbody>
</table>

It is clear that \( r(x) \) steadily decreases with \( x \) and \( h(x) \) is not always increasing.

We have postponed the physical interpretation available to some of the characteristic properties of the discrete models considered in Theorems 4.2 and 4.5 of the previous chapter for later consideration. It will now be established that they are in fact closely associated with the notion of being discussed in this section.

Theorem 5.8.

A necessary and sufficient condition for \( X \) to be DMRL is that

\[
V(Y_x) \leq r(x) [r(x)-1].
\]
The condition for IMRL class is obtained by reversing the inequalities.

Proof:

We have

\[ EY_x^2 = \frac{1}{R(x+1)} \sum_{y=x+1}^{\infty} (y-x)^2 f(x), \]

\[ = \frac{1}{R(x+1)} \sum_{1}^{\infty} n^2 f(x+n), \]

\[ = \frac{1}{R(x+1)} \sum_{1}^{\infty} n^2 [R(x+n)-R(x+n+1)], \]

\[ = 1 + \frac{1}{R(x+1)} \sum_{1}^{\infty} (2n+1) R(x+n+1). \] (5.4)

Also,

\[ EY_x^2 = \frac{1}{R(x+1)} \sum_{n=1}^{\infty} R(x+n), \]

\[ V(Y_x) = C_r(x) [r(x)-1]. \] (5.5)
Subtracting (5.5) from (5.4)

\[ EY_x^2 - r(x) = \frac{2}{R(x+1)} \sum_{n=1}^{\infty} n R(x+n+1), \]

where \( C \) is some positive constant. Taking \( C < 1 \), it follows that

\[
V(Y_x) - r(x) [r(x)-1] < 0.
\]

Conversely, we have shown in Theorem 4.5 that for every \( x > 0 \),

\[
V(Y_x) = C r(x) [r(x)-1]
\]

Therefore, for every \( x > 0 \),

\[
V(Y_x) < r(x) [r(x)-1].
\]
is equivalent to the statement

\[ r(x) - r(x+1) = \frac{1-C}{1+C}, \]

where \( C \) is some positive constant. Taking \( 0 < C < 1 \), it follows that

\[ V(Y_x) \leq r(x) [r(x)-1], \]

implies

\[ r(x) [r(x)-1] > 0. \]

Accordingly \( r(x) \) is a decreasing function of \( x \) and the theorem is completely proved.

Theorem 5.9.

\[ X \text{ is DMRL (IMRL) if } r(x) \text{ is not less than (not greater than) unity.} \]

Proof: From equation (4.10),

\[ 1-h(x+1) = \frac{r(x)-1}{r(x+1)}, \]

and therefore,

\[ r(x) = r(x+1) + \frac{(1-C)}{(1+C)}. \]

This implies

\[ h(x+1) \frac{r(x+1)}{r(x+1)-r(x)+1}. \]

For \( C > 1 \), we thus have \( r(x) < r(x+1) \) and hence MRL is an
Thus, $h(x+1) \leq 1$ implies $r(x+1) - r(x) \leq 0$ and consequently $X$ is DMRL.

The results of the last two theorems can be used to characterize the Waring and negative hypergeometric distributions among the class of distributions for which $h(x) r(x)$ is a constant or $b(x) = V(Y_x)/r(x)[r(x)-1]$ is a constant. We give the proof in one case only as the other follows by using the same argument.

Theorem 5.10.

Among the class of distributions with strictly increasing (decreasing) MRL, Waring (negative hypergeometric) is the only member for which $b(x)$ is a constant.

Proof:

The constancy of $b(x)$ gives rise to three cases, $b(x) \neq 1$, of which $b(x)=1$ does not provide a strictly increasing or decreasing MRL as it corresponds to the geometric law with constant MRL. Let us take $b(x)$ to be greater than unity. From Theorem 4.5 we find that $b(x)=C$ implies

$$r(x) = r(x+1) + (1-C)/(1+C).$$

For $C>1$, we thus have $r(x) \leq r(x+1)$ and hence MRL is an
increasing function of \( x \). Similarly, when \( C < 1 \), \( r(x) \) is decreasing in \( x \) and by Theorem 4.5 these values of \( C \) are characteristic of the Waring and negative hypergeometric distributions. This completes the proof.

5.5. Criteria Used in Maintenance Policies

Another category of ageing concepts considered in literature are those that help the study of maintenance policy which are followed to reduce the incidence of system failure or to return a failed system to the operating state. We consider some classes of distributions that are specially designed for application in this context. The continuous version of these classes are discussed in Marshall and Proschan (1972).

Definition 5.4.

A discrete random variable \( X \), with positive integer values as its support, or its distribution is new better than used or NBU if

\[
R(x+y+1) < R(x+1) R(y+1),
\]

for \( x, y > 0 \).
Definition 5.5.

X is new better than used in expectation or NBUE (new worse than used in expectation or NWUE) if

(a) X has finite (finite or infinite) mean m,

(b) \( r(x) < (>) m \), for \( x > 0 \).

The quantity \( R(x+y+1)/R(y+1) \) represents the survival function of a unit of age \( y \) or the conditional probability that a unit of age \( y \) will survive for an addition \( x \) unit of time. At \( y=0 \), \( [R(x+y+1)/R(y+1)] = R(x+1) \) is the survival function of a new unit and accordingly the ageing of the device can be studied by comparing \( R(x+y+1)/R(y+1) \) and \( R(x+1) \). Thus, \( R(x+y+1) \leq R(x+1) R(y+1) \) if and only if the older system has aged is that it has no better chance of surviving for a duration of \( x \) than does a new system. In other words, the new unit is better than the used one or NBU. A similar interpretation can be given to the NWU which exhibits the benefit of ageing. On the other hand the condition \( r(x) \leq r(o) = m \) states that the expected remaining life of a unit surviving age \( x \) is not larger than the expected life of a new unit so that the new unit fairs better than the unit of age \( x \) in terms of the expected life length. This explains the terminology NBUE. It is
easy to see that the boundary of both the classes is the geometric law,

\[ P[X=x] = q^{x-1}p, \quad x = 1, 2, \ldots, \]

\[ 0 < p < 1, \quad p+q = 1. \]

Definition 5.6.

The distribution of a positive integer valued random variable belongs to the HNBUE (harmonic new better than used in expectation) class if

\[
\sum_{t=x}^{\infty} R(t+1) \leq \sum_{t=x}^{\infty} G(t+1) = m(1 - \frac{1}{m})^x,
\]

where \( G(x) \) is the survival function of a geometric random variable with mean \( m = \sum R(x) \). The class of HNBUE distributions was introduced in the continuous case by Rolski (1975) in the above definition is as in Klefsjö (1982). It is obvious from the definition that the NBUE class is contained in the HNBUE class. We further have the following properties.

1. If \( X \) is NBU then \( X \) is NBUE.

This is obtained by summation of (5.6) with respect to \( y \) from 0 to \( \infty \).

2. \( X \) is NBU if and only if \( k(x) \geq k(o) \). (We use the definition of NBUE and the relationship \( k(x) = \frac{1}{r(x)} \).)
3. \( X \) is HNBUE if and only if \( \sum_{t=1}^{x} k(t) \geq k(o) \) for \( x=1,2,\ldots \).

As a consequence of Theorem 5.11 we can obtain a characterization of the geometric distribution as in the following theorem.

4. \( Y \) is NBUE if and only if \( h(o) \leq h(x) \).

5. \( Y \) is HNBUE if and only if \( h(o) \leq \sum_{t=1}^{x} h(t) \).

We have also,

Theorem 5.11.

For a non-decreasing function \( g(.) \), the random variable \( X \) is HNBUE if and only if

\[
Eg(Z) \geq Eg(Y), \quad (5.7)
\]

where \( Z \) is distributed as geometric with mean \( m \) and \( Y \) is the random variable in section 4.4.

Proof:

Since \( g \) is monotonic, the condition (5.7) is satisfied if and only if \( Z \succeq Y \). This means that

\[
( 1 - \frac{1}{m} )^{x-1} \geq \sum_{x=0}^{\infty} R(x+1),
\]

or

\[
m( 1 - \frac{1}{m} )^{x-1} \geq \sum_{x=0}^{\infty} R(x+1).
\]

Accordingly \( X \) is HNBUE.
As a consequence of Theorem 5.11 we can obtain a characterization of the geometric distribution as in the following theorem.

**Theorem 5.12.**

Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed random variables having \( E(X) = m < \infty \) with common survival function \( R(x) \). Then among the class of HNBUUE laws the geometric distribution,

\[
P[X = x] = pq^{x-1}, \ x = 1, 2, \ldots
\]

is the only law for which

\[
\frac{E(X_{(1)})^{-1}}{E(X_{(1)})} = \left( \frac{m-1}{m} \right)^n,
\]

where \( X_{(1)} = \min_{1 \leq i \leq n} X_i \).

**Proof:**

Assume \( X_i \) to be geometric with mean \( m \). Then,

\[
R(x) = (1 - \frac{1}{m})^{x-1}.
\]

Now,

\[
P[X_{(1)} \geq x] = [R(x)]^n
\]

\[
= [(1 - \frac{1}{m})^{n-1}]^{x-1},
\]
so that, \( X(1) \) is geometric with mean

\[
E(X(1)) = \frac{1}{1-(1-\frac{1}{m})^n}.
\]

Therefore,

\[
\frac{E(X(1))}{E(X_1)} - 1 = \left( \frac{m-1}{m} \right)^n,
\]

where,

\[
\frac{E(X(1))}{E(X_1)} = \sum_{x=1}^{\infty} \frac{l(x)f(x)}{p[X_1=x]},
\]

as stated.

Conversely, assume that \( X_i \) is HNBUE. Then we can write,

\[
E(X(1)) = \sum_{x=1}^{\infty} [R(x)]^n,
\]

we have from the HNBUE property of \( X_i \),

\[
\sum_{x=t+1}^{\infty} R(x) \frac{1}{m-1} \left[ R(x) \right]^{n-1},
\]

where,

\[
l(x) = \sum_{t=0}^{x} [R(t)]^{n-1}.
\]
Therefore,

$$E(X(1)) = R(1) \left[ l(1) - l(0) \right] + R(2) \left[ l(2) - l(1) \right] + \ldots ,$$

$$= l(1) \left[ R(1) - R(2) \right] + l(2) \left[ R(2) - R(3) \right] + \ldots ,$$

$$= \sum_{x=1}^{\infty} l(x) f(x) ,$$

where,

$$f(x) = P[X_i = x] ,$$

$$= R(x) - R(x+1) .$$

Defining,

$$h(x) = \sum_{t=0}^{x} (1 - \frac{1}{m})^{(n-1)t} ,$$

we have from the HNBUE property of $X_i$,

$$\sum_{x=t+1}^{\infty} R(x) \leq m \left( 1 - \frac{1}{m} \right)^{t} ,$$

$$= \sum_{t+1}^{\infty} (1 - \frac{1}{m})^{x} ,$$

or

$$\sum_{x=0}^{\infty} R(x) > \sum_{x=1}^{\infty} (1 - \frac{1}{m})^{x-1} .$$

Therefore,

$$0 \leq \sum_{x=0}^{\infty} R(x) .$$

where, $\phi(y) = m \left( 1 - \frac{1}{m} \right)^{n-1} (x-1)$. 
Also,

\[
\sum_{x=1}^{t} [R(x)]^2 - \sum_{x=1}^{t} \left(1 - \frac{1}{m}\right)^{2x} = \sum_{x=1}^{t} [R(x) - \left(1 - \frac{1}{m}\right)^{x-1}]R(x) + \sum_{x=1}^{t} [R(x) - \left(1 - \frac{1}{m}\right)^{x-1}] \left(1 - \frac{1}{m}\right)^{x-1} > 0.
\]

Proceeding similarly, we have,

\[
\sum_{x=1}^{t} [R(x)]^{n-1} > \sum_{x=1}^{t} \left(1 - \frac{1}{m}\right)^{(n-1)(x-1)}.
\]

In the above inequality, the equality sign will hold good if and only if \(h(x) = f(x)\), in which case,

Therefore,

\[
\ell(t) > h(t).
\]

Then,

\[
E(X_{(1)}) = \sum_{x=1}^{\infty} \ell(x)f(x) > \sum_{x=1}^{\infty} h(x)f(x),
\]

5.6 Conclusion

The various results and examples in the last three chapters constitute the properties of the class of discrete and continuous distributions that is parametrized by the linear mean residual life. The discrete version of the ageing concepts that parallel with their continuous counterparts are also been introduced. An elaborate study of these ageing concepts in the discrete time domain with

\[
\phi(Y) = m\left(1 - \frac{1}{m}\right)(n-1)(x-1).
\]
By Theorem 5.11,

\[ \mathbb{E}(X) \succ \mathbb{E}(Z), \]

\[ = \sum_{x=1}^{\infty} m(1 - \frac{1}{m})(n-1)(x-1) \frac{1}{m} \left(1 - \frac{1}{m}\right)^{x-1}, \]

\[ = \frac{1}{1 - (1 - \frac{1}{m})^{n}}. \]

In the above inequality, the equality sign will hold good if and only if \( \mathbb{I}(x) = h(x) \), in which case,

\[ R(x) = (1 - \frac{1}{m})^{x-1}, \]

as required in the Theorem.

5.6 Conclusion

The various results established in the last three chapters constitute the properties of the class of discrete and continuous distributions that are characterized by linear mean residual life. The discrete version of the ageing concepts that parallel with their continuous counterparts have also been introduced. An elaborate study of these ageing concepts in the discrete time domain with
respect to the preservation of these properties in relation to convolutions, mixing and coherent structures remains an open problem. The behaviour of the tail distributions point out to several applications in the analysis of income and reliability. Also, further characterizations of these models are to be investigated. The discussion in the present study is confined to univariate models only. It will be interesting to identify the multivariate models possessing appropriate multivariate analogues of the univariate properties discussed here. The answers to the various questions are being investigated and will be presented elsewhere.

REFERENCES


