Chapter IV

CHARACTERIZATION OF DISCRETE MODELS

4.1 Preliminaries

The aim of the present chapter is to extend to the discrete sample space some of the results that were established earlier to the Pareto, finite range and exponential models. As mentioned in the introduction, the distributions in this context centres around the three discrete distributions, Waring, negative hypergeometric and geometric which exhibit properties analogous to the continuous models just mentioned. Using the notations of section 2.4, we shall denote by \( W(a,b) \), the Waring distribution with probability function

\[
P[X=x] = \frac{(a-b)(b)^x}{(a)^{x+1}}, \quad x = 0, 1, 2, \ldots, \quad a > b > 0,
\]

while the symbol \( \text{NH}(k,n) \) is reserved for the negative hypergeometric law specified by,

\[
P[X=x] = \binom{k}{x}\binom{n-k}{n-x}, \quad x = 0, 1, 2, \ldots, n,
\]

and \( G(p) \) for the normal geometric model

\[
P[X=x] = q^xp, \quad x = 0, 1, 2, \ldots, \quad 0 < p < 1, \quad p+q=1.
\]
Notice that the discrete uniform distribution in the support of \((0,1,...,n)\) appears as a special case of (4.2) when \(k=1\).

4.2 Continuous Approximations

As a prelude to the explorations of characteristic properties of models (4.1), (4.2) and (4.3) in the rest of the chapter, a justification for alluding to them properties similar to the exponential, Pareto and finite range laws, as in order. One way of accomplishing this is by calculating the continuous approximation of the discrete distributions and thereby conclude that they lead to the desired continuous counter parts. The slope-ordinate ratio method propounded by Irwin (1975c) will be involved to realise our objective. Roughly speaking, the method consists in equating the ratio \(\frac{f_r - f_{r-1}}{f_r + f_{r-1}}\) to the logarithmic derivative of \(f(x)\) evaluated at \(x = r - \frac{1}{2}\) and then solve the resulting differential equation. Notice that \(f_r\) is the frequency for \(r=0,1,...\) and \(f(x)\) is the corresponding continuous density. For the Waring distribution \(W(a,b)\) of equation (4.1)

\[ f_r = N(a-b)(b)^r/(a)^{r+1}, \ r=0,1,2,..., \]

and we further deduce

\[ \frac{d}{dz} \log y = (b-a-1)/(a^2 - b + z), \ a>b. \]
so that
\[ f_r - f_{r-1} = N(a-b) \left[ \frac{(b)_r}{(a)_{r+1}} - \frac{(b)_{r-1}}{(a)_r} \right] , \]
\[ = N(a-b)(b-a-1) \frac{(b)_{r-1}}{(a)_{r+1}}, \]
and
\[ f_r + f_{r-1} = N(a-b) \left[ \frac{(b)_r}{(a)_{r+1}} + \frac{(b)_{r-1}}{(a)_r} \right], \]
\[ = N(a-b)(a+b+2r-1) \frac{(b)_{r-1}}{(a)_{r+1}}. \]

Hence,
\[ \frac{f_r - f_{r-1}}{\frac{1}{2}[f_r + f_{r-1}]} = \frac{2(b-a-1)}{(a+b+2r-1)} , \]
\[ \left[ f_r + f_{r-1} \right] = N \left[ k+n-r-2 \right] \frac{(k-1)}{n-r} / \left( k+n \right), \]
and this is called the slope-ordinate ratio at \( x = r - \frac{1}{2} \).

We now write
\[ \left( \frac{d}{dx} \log y \right)_{x=r-\frac{1}{2}} = \frac{2(b-a-1)}{(a+b+2r-1)} , \]
where \( y = f(x) \).

Setting \( z = x + \frac{1}{2} \), we have \( z=r \) whenever \( x = r - \frac{1}{2} \)
and we further deduce
\[ \frac{d}{dz} \log y = (b-a-1) \left( \frac{a-b-1}{2} + z \right), \ a>b. \]
The solution of the last differential equation is of the form

\[ y = f(x) = K(x^\alpha)^{-c-1}, \]

for some \( \alpha, c > 0 \), leading to \( P_{II}(c, \alpha) \).

Now consider the negative hypergeometric distribution with

\[ f_r = N \binom{-1}{r} \binom{-k}{n-r} / \binom{-1-k}{n}, \]

\[ = N \binom{k+n-r-1}{n-r} / \binom{k+n}{n}, \]

giving

\[ f_r - f_{r-1} = N \binom{k+n-r-2}{n-r-1} \binom{k-1}{n-r} / \binom{k+n}{n}, \]

and

\[ \frac{1}{2}[f_r + f_{r-1}] = N \binom{k+n-r-2}{n-r-1} \binom{k-1}{n-r} + n-r / \binom{k+n}{n}. \]

Thus

\[ \frac{d}{dz} \log y = (k-1) / ((k-1)/2 + n-z) \]

or

\[ h(y) = f(x)/R(x), \]

\[ f(z) = \text{const} \ (1- z/R)^d-1, \]

where \( d \) and \( R = k + n - \frac{k}{2} \) are both positive. This shows that the finite range model is the continuous approximation of the negative hypergeometric law.
where \( d = K \) and \( R = \frac{k}{2} + n - \frac{1}{2} \) are both positive. This shows that the finite range model is the continuous approximation of the negative hypergeometric law.

4.3 Failure Rate and MRL

The distributions that are to follow require elaborate use of the concepts of failure rates and MRL's as applied to discrete random variables. These concepts have been touched upon in many works such as Kalbfleth and Prentice (1980). Many interrelationships and identities in this connection will be investigated now.

Let \( X \) be a discrete random variable in the support of \( I^+ = (0,1,2,\ldots) \) with probability mass function \( f(x) \). We also define

\[
R(x) = P[X \geq x],
\]

so that

\[
f(x) = P[X=x] = R(x) - R(x+1) .
\]

The failure rate of \( X \) is

\[
h(x) = f(x)/R(x),
\]

and

\[
f(x+1) = f(x) h(x+1) (1-h(x))/h(x).
\]
and the MRL is

\[ r(x) = \mathbb{E}[X - x | X > x], \]

\[ = \frac{1}{R(x+1)} \sum_{y=x+1}^{\infty} (y-x) f(y). \] (4.9)

This gives

\[ r(x) R(x+1) = \sum_{x+1}^{\infty} R(y), \] (4.6)

and the recurrence relation

\[ r(x) R(x+1) - r(x+1) R(x+2) = R(x+1), \] (4.7)

or

\[ r(x) R(x+1) = [r(x-1)-1] R(x), \] (4.7)

From the equation (4.5) and (4.4)

\[ h(x) = \frac{[R(x)-R(x+1)]}{R(x)}, \]

\[ R(x) = \prod_{u=0}^{x-1} \frac{1-f(u)}{1-f(x)}. \] (4.11)

Hence,

\[ 1-h(x) = \frac{f(x+1)/h(x+1)}{f(x)/h(x)}, \]

and

\[ f(x+1) = f(x) h(x+1) (1-h(x))/h(x). \]
Thus we get by iteration on $x$

$$f(x) = h(x) (1-h(x-1)) \ldots (1-h(o)),$$  \hspace{1cm} (4.8)

and

$$R(x) = \prod_{y=o}^{x-1} (1-h(y)).$$  \hspace{1cm} (4.9)

Equations (4.8) and (4.9) show that $h(x)$ determines the distribution of $X$ uniquely.

Combining (4.7) and (4.9) we get the relationship between the failure rate and MRL of $X$ as

$$1-h(x+1) = \frac{r(x)-1}{r(x+1)}, \hspace{1cm} x>0.$$  \hspace{1cm} (4.10)

It follows that MRL function also determines the distribution uniquely through

$$R(x) = \prod_{u=1}^{x-1} \frac{r(u)-1}{r(u)} (1-f(o)).$$  \hspace{1cm} (4.11)

These interrelationships are useful in lifelength studies when time is treated as discrete as explained in chapter V but their immediate application is restricted to characterizing probability distributions.

* These results have been published in Prob.Statist.Letters (reference 65 ).
4.4 Characterizations By Distribution Based on Partial Sums*

If X is a random variable defined in the previous section with \( E(X) = m < \infty \), the variate Y specified by

\[
g(y) = P[Y=y] = m^{-1} P[X>y], \quad y=0,1,\ldots,
\]

(4.12)
is said to have the distribution based on partial sums corresponding to X. The probabilities assumed by the values of Y are proportional to the survival probabilities of X. Some properties of (4.12) are discussed in Johnson and Kotz (1969, p.261). Under certain conditions Gupta (1979) describes Y as the residual life time of a component, in a system where a component of life length X is replaced upon failure by another, having the same life distribution, so that the sequence of life lengths forms a renewal process. He showed that the failure rate of Y is the reciprocal of the MRL of X and that when the renewal distribution belongs to the class of modified power series distribution, the geometric law is the only one satisfying the property \( E(X)=E(Y) \). In this section we supplement Gupta's results by extending some of his results to cover the class of discrete distributions under consideration and also explore the possibility of arriving at some new characterizations.

* These results have been published in Prob.Statist.Letters (reference 65 ).
4.4.1 Basic Results

Analogous to equation (4.5), we write the failure rate of $Y$ as

$$k(s) = \frac{P[Y=s]}{P[Y>s]},$$

$$= \frac{R(s+1)}{\sum_{s+1}^{\infty} R(u)},$$

$$= [r(s)]^{-1}, s > t > 0,$$

(4.13)

where $r(s)$ is as in equation (4.6), the MRL of $X$. Result (4.13) is obtained by Gupta (1979). Further,

$$k(s) = \sum_{r=1}^{\infty} \frac{P[Y=s+r]}{\sum_{s+1}^{\infty} R(u)},$$

$$= \sum_{r=1}^{\infty} \frac{R(s+r) h(s+r)}{\sum_{s+1}^{\infty} R(u)},$$

$$= \sum_{s+1}^{\infty} h(t) w(t),$$

(4.14)

with

$$w(t) = \frac{R(t)}{\sum_{s+1}^{\infty} R(s), t = s+1, s+2, \ldots}.$$

Thus the failure rate of $Y$ is a weighted average of the failure rates of $X$ beyond the time point $s$. From (4.14),
Thus we see that, as in the case of failure rate, the MRL for \( G(u) = P[Y > u] \),

\[
a(s) = \sum_{s+1}^{\infty} \frac{G(t)}{G(s+1)}, \tag{4.15}
\]

so that together with (4.12) giving

\[
G(s) = m^{-1} \sum_{s+1}^{\infty} R(t), \tag{4.17}
\]

we can write,

\[
\sum_{s+1}^{\infty} \frac{G(t)}{G(s+1)} = m^{-1} \sum_{s+1}^{\infty} R(t).
\]

Substituting

\[
r(s) R(s+1) = \sum_{s+1}^{\infty} R(t),
\]

the terms in (4.16) can be simplified to

\[
w(t+u+1) r(u+t) = \begin{cases} 1, & u=1, \\ \frac{u}{\prod_{j=1}^{u} [1-k(t+1+j)]}, & u=2,3,\ldots. \end{cases}
\]

the terms in (4.16) can be simplified to
Thus we see that, as in the case of failure rate, the MRL of Y is also a weighted average of the MRL's of X.

Now with the aid of (4.13), equation (4.10) becomes,
\[ 1-h(s+1) = [(k(s))^{-1}-1] k(s+1), \]
or
\[ h(s) = 1+k(s)[1-(k(s-1))^{-1}], \quad s \geq 1, \quad (4.17) \]
giving the relationship between failure rate of X and Y.

On the other hand, the MRL's of X and Y satisfy
\[ r(s+1) = a(s+1) \left[1+a(s+1)-a(s)\right]^{-1}, \]
which follows from (4.17), (4.13) and (4.10). We can also have the equation
\[ k(s) = \frac{[1+a(s)-a(s-1)]}{a(s)}, \]
connecting the failure rate of Y and the MRL of X.

4.4.2 Characterizations

An immediate consequence of the identities developed in the previous section is in characterizing some discrete distributions. We first prove...
Theorem 4.1.

A necessary and sufficient condition for $X$ to be distributed as $G(p)$ $(W(a,b); \text{NH}(k,n))$ is that $Y$ is $G(p)$ $(W(a,b+1); \text{NH}(k+1, n-1))$.

Proof:

From Xekalaki (1983a), it is seen that a failure rate function of the form $h(x) = (L+Mx)^{-1}$ characterizes $G(p)$ for $M = 0, L = p^{-1}$; $W(a,b)$ for $M = (a-b)^{-1}, L = a(a-b)^{-1}$; and $\text{NH}(k,n)$ for $M = -(k)^{-1}, L = k^{-1}(n+k)$.

Substituting this form of $h(x)$ in (4.17), we find,

$$1+k(x)(1-(k(x-1))^{-1}) = (L+Mx)^{-1}.\quad (4.18)$$

Since, $k(x) = (r(x))^{-1}$, this should read

$$\frac{1-r(x-1)}{r(x)} = (L+Mx)^{-1} - 1.\quad (4.20)$$

Applying the above result recursively for ascending values of $x$, the parameters of each model after inserting the values

The form of $k(x)$ suggests that the distribution of $Y$ of the same type as that of $X$ and further (4.20) generalizes the original (4.17).
\[ r(x-1) = \sum_{\gamma=0}^{\infty} \frac{(\alpha+x)(\gamma)}{(\beta+x)(\gamma)}, \quad (4.18) \]

with

\[ \alpha = \frac{(L-1)}{M} \text{ and } \beta = \frac{L}{M}. \]

Obviously the last equation represents the hypergeometric series \( \text{2F1}(\alpha+x, 1, \beta+x, 1) \) in the usual notations (see Abramowitz and Stegun, 1972). From the well known formula,

\[ \text{2F1}(p,q,y,1) = \frac{\Gamma(y-p-q)\Gamma(y)}{\Gamma(y-p)\Gamma(y-q)}, \]

it is easy to see that

\[ \text{2F1}(p,1,y,1) = \frac{(y-1)}{(y-p-1)}. \quad (4.19) \]

Thus

\[ r(x-1) = \frac{(\beta+x-1)}{(\beta-\alpha-1)}, \quad (4.20) \]

or

\[ k(x) = (A+Bx)^{-1}, \]

where

\[ A = L(1-M)^{-1} \text{ and } B = M(1-M)^{-1}. \quad (4.21) \]

The form of \( k(x) \) suggests that the distribution of \( Y \) is of the same type as that of \( X \) and further (4.20) gives the parameters of each model after inserting the values

\[ r(x) = \frac{(\beta+x)}{(\beta-\alpha-1)}, \quad (4.21) \]
of L and M for each stated at the beginning of the proof.

Note:

The fact that the above models are closed with respect to the formation of the distribution of Y can also be established by direct calculation using the equation (4.12). Our motivation in using the above method are (i) direct calculation are more extensive, (ii) the form of the failure rate and MRL is handy when the modelling is based on these concepts, and (iii) the relationships used here are important in their own right in other areas of applications as will be shown in chapter V.

Corollary 4.1.

The MRL of X is of the form \(A+Bx\) if and only if

- \(X\) is either \(G(p)\) for \(A=0\) or \(W(a,b)\) for \(A > 0\) or \(NH(k,n)\) for \(A < 0\).

Proof:

When \(X\) has a distribution that follows one of the models stated in the theorem if and only if the failure rate is of the form \((L+Mx)^{-1}\). In this situation, from equation (4.20) we see that

\[ r(x) = \frac{(\beta+x)}{(\beta-\alpha-1)}, \quad (4.22) \]
and therefore, the MRL is of the required form. Conversely, if MRL is as in (4.22), the failure rate is

\[ h(x) = 1 - \left( \frac{r(x-1)}{r(x)} \right)^{-1} \]

\[ = (\beta - \alpha) / (\beta + x) = (L+Mx)^{-1} \]

Accordingly, by the characterization theorem of Xekalaki (1983a), X has the above specified distributional form.

To enable future reference, we note Table 4.1 for the actual expressions for the failure rates and MRLs of the various models.

**Table 4.1.**

Failure rates and MRL's of Discrete Models

<table>
<thead>
<tr>
<th>Model</th>
<th>Failure Rate</th>
<th>MRL</th>
</tr>
</thead>
<tbody>
<tr>
<td>G(p)</td>
<td>p</td>
<td>p^{-1}</td>
</tr>
<tr>
<td>W(a,b)</td>
<td>(a-b)(a+x)^{-1}</td>
<td>(a+x)(a-b-1)^{-1}</td>
</tr>
<tr>
<td>NH(k,n)</td>
<td>k(k+n-x)^{-1}</td>
<td>(k+n-x)(k+1)^{-1}</td>
</tr>
<tr>
<td>(Uniform in [0,n])</td>
<td>(1+n-x)^{-1}</td>
<td>1/2 (1+n-x)</td>
</tr>
</tbody>
</table>
Theorem 4.2

The property \( h(x) = C \cdot k(x) \) for all integers \( x \geq 0 \) and a constant \( C \), characterizes \( G(p) \) for \( C=1 \), \( W(a,b) \) for \( C > 1 \) and \( NH(k,n) \) for \( 0 < C < 1 \).

Proof: This follows from Theorem 4.2 and the relationship \( h(x) = C \cdot k(x) \) for all integers \( x \geq 0 \) and a constant \( C \) if and only if \( X \) is \( G(p) \) for \( C=1 \), \( W(a,b) \) for \( C > 1 \) and \( NH(k,n) \) for \( 0 < C < 1 \).

For the geometric law \( X \) and \( Y \) have identical distributions with the same parameters and therefore, in this case \( h(x) = k(x) \). When \( X \) is \( W(a,b) \), \( Y \) is \( W(a,b+1) \) so that from table 4.1,

\[
\frac{h(x)}{k(x)} = \frac{a-b}{a-b-1} > 1.
\]

In the negative hypergeometric case

\[
\frac{h(x)}{k(x)} = \frac{k}{k+1} < 1.
\]

Conversely, \( h(x) = C \cdot k(x) \) is equivalent to \( h(x) \cdot r(x) = C \) or to

\[
r(x) - r(x-1) + 1 = C.
\]

The solution of this equation is

\[
r(x) = (C-1)x + r(0),
\]

which is of the form \( A+Bx \) and therefore by Corollary 4.1 our Theorem is proved.
Corollary 4.2.

The MRL and failure rate of $X$ is such that

$$r(x) h(x) = C \text{ for all integers } x > 0 \text{ and a constant } C > 0$$

if and only if $X$ is $G(p)$ for $C=1$, $W(a,b)$ for $C > 1$ and $NH(k,n)$ for $0 < C < 1$.

This follows from Theorem 4.2 and the relationship $k(x) = [r(x)]^{-1}$.

Corollary 4.3.

The relationship $r(x) = Ka(x)$ is satisfied for all integers $x > 0$ and a constant $K$ if and only if $X$ is geometric (for $K=1$) or Waring (for $K>1$) or negative hypergeometric (for $K>1$).

Proof:

The result follows from the relationships between $a(x)$ and $k(x)$, equation (4.17) and Theorem 4.2.

The utility of these results and the physical interpretation of the properties enjoyed by the models in the last theorem in the context of ageing will be taken up in chapter V. The advantages arising out of these Theorems in the context of evaluating the memory of discrete distributions will now be explored.
A measure of memory at any point $x$ in the support $(0,1,2,\ldots)$ of a discrete random variable $X$ is defined in terms of MRL (Nair, 1983) as

$$m(x) = r(x-1) - r(x), \quad x \geq 1.$$ 

The distribution of $X$ is said to have no memory, negative memory or positive memory according as $m(x)$ is zero, negative or positive. Since a distribution can have different types of memory at the various points of its support, a consolidated measure of memory for the entire support was obtained as a weighted average of the measures at various points. The proposed measure was

$$M = \frac{2E^2(X) + E(X) - E(X^2)}{E(X^2) + E(X)}. \quad (4.23)$$

The distribution itself has lack of memory, negative memory or positive memory according as $M$ is zero, negative or positive. The following Theorems follow from the above definition.

Theorem 4.3.

The geometric, Waring and negative hypergeometric laws in that order are the only discrete distributions that possess lack of memory, constant negative memory and constant positive memory at each point of its support.
(This deduction is easily seen from the fact that $r(x) = A + Bx$ for these models).

Theorem 4.4.

If $X$ has a particular type of memory, at a given point, then $Y$ also has the same type of memory.

Proof:

Suppose $X$ has positive memory at the point $x$.

This implies that

$$r(x-1) > r(x) \text{ or } k(x) > k(x-1).$$

The last inequality, however, is equivalent to

$$\frac{g(x) G(x-1) - g(x-1) G(x)}{G(x)} > 0,$$

or to

$$\frac{G(x)}{G(x-1)} > \frac{G(x+1)}{G(x)}.$$

Hence $H(x) = G(x)/G(x-1)$ is decreasing in $x$.

Accordingly,

$$a(x) = H(x) + H(x) H(x+1) + H(x) H(x+1) H(x+2) + \ldots,$$

and

$$a(x) - a(x+1) = [H(x) - H(x+1)] + H(x+1) [H(x) - H(x+2)] + \ldots > 0,$$
which means that $Y$ has positive memory at $x$. The proof is similar when $X$ has lack of memory or has negative memory.

From the expression for $M$ it is clear that, in general, $M=0$ does not imply that $X$ is geometric (although for this law $M=0$). This means that we have to restrict the family of distributions to be considered, in order to characterize the geometric law by the property $M=0$.

Suppose that $X$ belongs to the modified power series family

$$P[X=x] = a(x) [g(\Theta)]^x/f(\Theta), \ x \in B,$$

where $B$ is a subset of the set of non-negative integers, $a(x) > 0$, $g(\Theta)$ and $f(\Theta)$ are positive, finite and differentiable. Then from Gupta (1979), we have $E(Y) = E(X)$.

If we denote the generating function of $\langle f(x) \rangle$ and $\langle R(x) \rangle$ by

$$A(t) = \Sigma f(x) t^x \quad \text{and} \quad B(t) = \Sigma R(x) t^x,$$

it is easy to see that

$$A'(1) = E(X) \quad \text{and} \quad A''(1) = EX(X-1). \quad (4.24)$$
Further,

\[ (1-t)B(t) = 1 - tA(t). \]  (4.25)

Differentiating (4.25) twice, with respect to \( t \) and setting \( t=1 \),

\[ B'(1) = \sum_{x=1}^{\infty} xR(x), \]

and

\[ 2B'(1) = A''(1) + 2E(X), \]

\[ = E(X) - 1 + 2E(X). \]

Accordingly,

\[ E(X) = E(Y) = m^{-1} \sum yP[Y \geq y], \]

\[ P(y|X) = P(x < X < x+y|X > x), \]

\[ = B'(1) - E(X). \]

Thus,

\[ E(X) = \frac{1}{2} m^{-1}[E(X^2) - E(X)], \]

\[ = \frac{[E(X^2) - E(X)]}{2E(X)}, \]

or

\[ E(X^2) = 2E^2(X) + E(X), \]

and hence \( M = 0 \). We have, therefore, established the following result.
Theorem 4.5.
Among the modified power series family, geometric law is the only one for which $M=0$.

4.5 Residual Life Distribution

Let $X$ be a random variable representing the life time of a component or device in the support of the set of non-negative integers with survival function $R(x)$. When the life times are expressed only in completed units of time, the domain of $X$ will be restricted to the support just mentioned. In such a situation, the residual life distribution (RLD) of $X$ at the elapse of $x$ units of time is specified by the distribution function

$$F(y;x) = P[x < X < x+y|X > x],$$

$$= \frac{F(x+y) - F(x)}{1-F(x)},$$

$$= \left[\frac{R(x+1) - R(x+y+1)}{R(x+1)}\right].$$

The corresponding survival function is

$$R(y;x) = \frac{R(x+y+1)}{R(x+1)}, \quad y > 0.$$ \hspace{1cm} (4.26)

For convenience, let $Y_x$ denote the random variable with
survival function (4.26). Then the MRL of $x$ is from (4.6)

$$r(x) = \frac{1}{R(x+1)} \sum_{z=x}^{\infty} (z-x) f(z),$$

where $f(.)$ is the probability mass function of $X$. This can be written as

$$r(x) = \frac{1}{R(x+1)} \sum_{z=x}^{\infty} R(z),$$

so that the definition (4.26) is consistent with the notion of MRL and residual life given in section 4.4.

Our first concern is the form of the RLD when $X$ follows the class of models considered so far. This is vindicated through the following theorem.

Theorem 4.6.

$Y_x$ is $G(p)$ (or $W(a+x+1, b+x+1)$ or $NH(k, n-x-1)$) if and only if $X$ is $G(p)$ (or $W(a, b)$ or $NH(k, n)$) and conversely.
Proof:

When $X$ follows $G(p)$, $R(x) = q^x$ and hence,

$$R(y; x) = \frac{q^{x+y+1}}{q^{x+1}} = q^y = R(y). \quad (4.27)$$

Then the RLD is also geometric with the same parameter as $X$. On the other hand, $X$ is $W(a, b)$ implies,

$$R(x) = (a-b) \sum_{x}^{(b)_t/(a)_{t+1}},$$

Accordingly,

$$= (a-b) \frac{(b)^x}{(a)^{x+1}} \sum_{x=0}^{\infty} \frac{(b+x)_{r}/(a+x+1)_{r},}{(a-b)} \quad (from \ formula \ (4.19))$$

$$= (b)_x/(a)_x. \quad (4.28)$$

Accordingly,

$$R(y; x) = \frac{(b)^{x+y+1} (a)^{x+1}}{(a)^{x+y+1} (b)^{x+1}},$$

$$= \frac{(b+x+1)^{y}}{(a+x+1)^{y}}, \quad y = 0, 1, 2, \ldots \quad (4.29)$$

and therefore $Y_x$ is $W(a+x+1, b+x+1)$. Lastly when $X$ is $NH(k, n)$.
\[ R(x) = \sum_{x} \frac{(-1)^{x} (n-x)}{n} / \binom{n}{n}, \]

\[ R(y; x) = \sum_{x} \frac{(k+n-x-y-1)}{n-x-y-1} / \binom{n-x-1}{n-x}, \]

and therefore from (4.36),

\[ \frac{R(x+y+1)}{R(x+1)} = \frac{(b+x+1)}{(a+x+1)}. \]

Thus,

\[ R(x) = \sum_{x=0}^{n-x} \frac{(-1)^{x} (k-t-1)}{k+n} / \binom{n-x}{n}. \]

The last sum is, however, reduced to the following by virtue of the combinatorial identity (Riordan, 1968)

\[ \sum_{x=0}^{n} \binom{a+n-x-1}{n-x} = \binom{a+n}{n}, \quad (4.30) \]

so that,

\[ R(x) = \frac{(k+n-x)}{n-x} / \binom{n-x}{n}, \]

and

\[ R(y; x) = \frac{(k+n-x-y-1)}{n-x-y-1} / \binom{n-x-1}{n-x-1}, \]

showing that \( Y_x \) is NH\((k,n-x-1)\).
Let us look at the converse proposition. Here we are given that, for example, in the Waring case,

\[ R(y;x) = \frac{(b+x+l)y}{(a+x+l)y}, \]

and therefore from (4.26),

\[ \frac{R(x+y+1)}{R(x+1)} = \frac{(b+x+1)y}{(a+x+1)y}. \]

Setting \( x \) to zero,

\[ \frac{R(y+1)}{R(1)} = \frac{(b+1)y}{(a+1)y}, \]

and hence,

\[ R(y) = \frac{(b)y}{(a)y}. \]

The proof for other models are on similar lines and the truth of the theorem is established.

4.6 Characterization By Properties Of Residual Life*

In continuation with the characterization of the models by properties of mean residual life renewed in

* The results in this section have appeared in Cal. Statist. Assoc. Bull. (Reference 33)
chapter 2, we now present certain properties based on the form of the variance of residual life. The applications of the Theorem in reliability analysis is discussed in chapter V.

Theorem 4.7.

If \( b(x) = \frac{\text{V}(Y(x))}{\text{EY}(x)} \cdot \text{E}[Y(x)-1] = C \), a constant, then a necessary and sufficient condition that \( X \) follows

(i) geometric distribution is \( C=1 \)

(ii) negative hypergeometric distribution, \( \text{NH}(k,n) \), is \( C < 1 \), and

(iii) Waring distribution, \( \text{W}(a,b) \), is \( C > 1 \).

Note that the symbol \( \text{V}(X) \) stands for the variance of the random variable \( X \).

Proof:

First we prove that the condition is necessary.

By definition,

\[
\text{V}[Y(x)] = Cr(x) \cdot (r(x)-1),
\]

where,

\[
\text{V}[Y(x)] = \frac{1}{R(x+1)} \sum_{x+1}^{\infty} (y-x)^2 f(y) - r^2(x),
\]
or,

\[ r(x) R(x+1)[Cr(x)-C+r(x)]= \sum_{n=1}^{\infty} n^2 f(x+n), \]

we find,

\[ = \sum_{n=1}^{\infty} n^2[R(x+n)-R(x+n+1)], \]

which is the same as

\[ = R(x+1)+ \sum_{n=1}^{\infty} [(n+1)^2-n^2] R(x+n+1). \]

The right hand side may be simplified as,

\[ = R(x+1)+2 \sum_{n=1}^{\infty} nR(x+n+1)+ \sum_{n=1}^{\infty} R(x+n+1)= \sum_{n=1}^{\infty} R(x+n)+2 \sum_{n=1}^{\infty} nR(x+n+1), \]

This, however, reduces to,

\[ r(x)=r(x+1)+(1-x)/(1+e). \]  

Notice that when \( C=1, \)

\[ 2 \sum_{n=1}^{\infty} nR(x+n+1) = r(x)R(x+1)(C+1)(r(x)-1). \]  

(4.31)

and this implies that \( r(x)=k, \) a constant. From the

Changing \( x \) to \( x+1 \) in (4.31) and subtracting the resulting

expression from (4.31), we get

\[ 2[R(x+2)-R(x+3)+2R(x+3)-2R(x+4)+ ... ] \]

\[ = (C+1)[r(x) R(x+1)(r(x)-1)-r(x+1)R(x+2)(r(x+1)-1)]. \]
On using the recurrence relation,

\[ [r(x)-1] R(x+1) = r(x+1) R(x+2), \]

we find,

\[ \sum_{n=1}^{\infty} R(x+n+1) = (C+1) [r(x)-r(x+1)+1] r(x+1) R(x+2), \]

which is the same as

\[ 2r(x+1)R(x+2) = (C+1)r(x+1)R(x+2)[r(x)-r(x+1)+1]. \]

This, however, reduces to,

\[ r(x) = r(x+1) + \frac{1-C}{1+C}. \quad (4.32) \]

Notice that when \( C=1 \),

\[ r(x) = r(x+1), \text{ for all } x > 0, \]

and this implies that \( r(x) = K \), a constant. From the definition of \( r(x) \), \( K \) is greater than unity so that there exists a \( p \), satisfying \( 0 < p < 1 \) such that \( K = p^{-1} \). Hence from (4.11),

\[ R(x) = (1-f(0)) q^{x-1}, \quad q=1-p. \]

Determining \( f(0) \) such that \( R(0)=1 \), we get,

\[ f(x) = pq^x, \quad x = 0, 1, 2, \ldots, \]

and \( X \) has geometric distribution as claimed.
Now taking $C<1$, from equation (4.32), we find
\[ r(x+1) = r(x) + m, \]
so that $r(x)$ is of the form $l + mx$ where $l > 1$ and $m < 0$
and therefore $X$ is $\mathcal{W}(a,b)$.

Lastly, when $C>1$, $r(x) = l + mx$ with $l > 1$, but $m > 0$
so that by applying corollary 4.1 we conclude that $X$ is $\mathcal{NH}(k,n)$.

It remains to prove the sufficiency of the conditions of the Theorem. We use the formula (4.31) to get,
\[ (C+1)r(x)[r(x)-1] = 2[R(x+1)]^{-1} \sum_{n=1}^{\infty} nR(x+n+1), \]
\[ Cr(x) (r(x)-1) = 2 s(x) - r(x) [r(x)-1], \]
and
\[ V(Y(x)) = 2s(x)-r(x) (r(x)-1), \]
where,
\[ s(x) = (R(x+1))^{-1} \sum_{n=1}^{\infty} nR(x+n+1), \]
\[ b(x) = 2s(x) [r(x) (r(x)-1)]^{-1} -1. \quad (4.33) \]
When $X$ is geometric, $r(x)=p^{-1}$, $s(x) = qp^{-2}$, so that $b(x)=1$. For the Waring distribution in (4.1),
Finally, we write \( b(x) \) as

\[
= \sum_{i=2}^{\infty} \frac{(b)_{x+i}}{(a)_{x+i-1}} \left\{ \frac{1}{(a+x+i-1)} + \frac{(b+x+i)}{(a+x+i-1)(a+x+i)} + \ldots \right\},
\]

\[
b(x) = \frac{b(b+1)_{x+i-1}}{(a)_{x+i-1}} \left\{ \frac{1}{a-b-1} \right\}, \quad a > b+1,
\]

\[
= \frac{b}{a-b-1} \left\{ \frac{(b+1)_{x+1}}{(a)_{x+1}} + \frac{(b+1)_{x+2}}{(a)_{x+2}} + \ldots \right\},
\]

\[
x(x) = \frac{b}{a-b-1} \left\{ \frac{1}{(a+x)} + \frac{(b+x+2)}{(a+x)(a+x+1)} + \ldots \right\},
\]

\[
= \frac{b}{a-b-1} \frac{(b+1)_{x+1}}{(a)_{x}} \frac{1}{a-b-2}, \quad a > b+2.
\]

Thus,

\[
R(x+1)s(x) = \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} \frac{(b)_{x+j}}{(a)_{x+j}}, \quad (4.34)
\]
Also, from table 4.1,
\[ r(x) = \frac{a+x}{a-b-1}. \]

Finally, we write \( b(x) \) as
\[
b(x) = \frac{2(b+x+1)(a+x)(a-b-1)^2}{(a-b-1)(a-b-2)(a+x)(b+x+1)} - 1,
\]
\[= -1 + 2(a-b-1)/(a-b-2) > 1. \]

To compute the expression (4.32) for the negative hypergeometric distribution, observe that its probability mass function (4.2) can be converted into the form
\[
f(x) = \binom{k+n-x-1}{n-x} / \binom{k+n}{n},
\]
resulting in the survival function,
\[
R(x) = \binom{k+n-x}{n-x} / \binom{k+n}{n}.
\]

Accordingly from (4.34),
\[
\binom{k+n-x-1}{n-x-1} s(x) = \sum_{i=2}^{n-i} \sum_{y=0}^{a+n-y} \binom{k+\gamma-y-1}{n-y-1}, \quad m=n-i,
\]
\[
= \binom{k+n-x-1}{n-x-1} / \binom{k+n-y-1}{n-y-1} - 1.
\]
on using the combinatorial identity (4.30).

\[
\begin{align*}
\text{b}(x) &= \frac{2(k+n-x)(n-x-1)(k+1)^2}{(k+1)(k+2)(k+n-x)(n-x-1)} - 1, \\
\text{Thus,}
\end{align*}
\]

\[
\begin{align*}
(k+n-x-1) s(x) &= \sum_{y=0}^{n-2} \binom{k+1+n-y-2-x}{n-y-2-x} = \binom{k+n-x}{n-x-2},
\end{align*}
\]
or

\[
\begin{align*}
s(x) &= (k+n-x)(n-x-1)/(a+1)(a+2).
\end{align*}
\]

4.7. **Characterization by Additive Damage Model**

Again, we have

\[
\begin{align*}
r(x) &= \frac{1}{R(x+1)} \sum_{y=x+1}^{n} R(y),
\end{align*}
\]

\[
\begin{align*}
&= \binom{k+n-x-1}{n-x-1}^{-1} \sum_{y=x+1}^{n} \binom{a+n-y}{n-y}, \\
&= \binom{k+n-x-1}{n-x-1}^{-1} \sum_{y=0}^{n-x-1} \binom{k+n-y-x-1}{n-y-x-1},
\end{align*}
\]

for some positive integer \( t \). We further assume that

\[
\begin{align*}
E[U|X=t] &= k+m x, \quad l \geq 0, \quad m \neq 0.
\end{align*}
\]
Thus, \( Y \) becomes the damaged component of \( X \) where, the random mechanism that relates \( X \) to \( Y \) is represented by the regression function \( \hat{Y} \) on \( X \). If we assume that the conditional expectation is a different linear function of the form,

\[
E[Y|X=x] = \alpha + \beta x,
\]

then it is possible to arrive at a characterization of our result.

**Theorem 4.8.**

**Given that**

\[
b(x) = \frac{2(k+n-x)(n-x-1)(k+1)^2}{(k+1)(k+2)(k+n-x)(n-x-1)} - 1,
\]

\[
= \frac{2(k+1)}{(k+2)} - 1 = \frac{k}{k+2} < 1.
\]

This completes the proof.

### 4.7. Characterization By Additive Damage Model

Let \( X, Y \) and \( U \) denote random variables in the support of the set of non-negative integers such that for \( X \) to be distributed as geometric for \( \theta = \frac{m}{\lambda} \) and \( \lambda > 0 \), \( m > 0 \), we have:

\[
Y = X - U, \quad 0 < U < \max(0, X-t) \quad \text{for some positive integer } t.
\]

We further assume that for \( x > t, t > 0 \),

\[
E[U|X=x] = \lambda + mx, \quad \lambda > 0, \quad m \neq 0.
\]
Thus, \( Y \) becomes the damaged component of \( X \) where, the random mechanism that reduces \( X \) to \( Y \) is represented by the regression function \( U \) on \( X \). If we assume that the conditional mean of \( U \) given \( X > x \) is a different linear function of the form

\[
E[U|X > x] = \alpha + \beta x,
\]
then it is possible to arrive at a characterization of our discrete models.

**Theorem 4.8.**

Given that

\[
E[U|X = x] = l + mx,
\]

it is necessary and sufficient that

\[
E[U|X > y] = \alpha + \beta y,
\]

for \( X \) to be distributed as geometric for \( \beta = m > 0 \) and \( l > 0 \); Waring for \( \beta > m > 0 \) and \( \alpha > l \); negative hypergeometric for \( \beta < m < 0 \) and \( \alpha > l \).

Conversely, if \( E(U|X > y) \) is of the form in (4.36),
Proof:

We have

\[ E[U|X>y] = \frac{1}{R(y+1)} \sum_{y+1}^{\infty} (l+mx) f(x), \]

so that

\[ \tau(y) = Ay + B \]

with

\[ A = \frac{\beta}{\alpha + \beta - 1} \]

and

\[ B = \frac{\alpha - 1}{\alpha + \beta - 1}. \]

The form of the distribution of \( X \) follows from (4.37), corollary 4.1 to Theorem 4.1.

From table 4.1, we read the values of \( r(y) \), to find,

\[ E[U|X>y] = \begin{cases} 
(\lambda + mp^{-1}) + my, & \text{for } G(p), \\
\lambda + am(a-b-1)^{-1} + \frac{m(a-b)y}{(a-b-1)}, & \text{for } W(a,b), \\
\lambda + \frac{(k+n)m}{(k+1)} + \frac{mk}{(k+1)} y, & \text{for } NH(k,n),
\end{cases} \]

which is of the form \( \alpha + \beta y \). The conditions on the parameters \( \alpha \) and \( \beta \) can easily verified to be as stated in the Theorem.

Conversely, if \( E(U|X>y) \) is of the form in (4.36),
we have from (4.37),

$$\alpha + \beta y = l + my + mr(y)$$

so that

$$r(y) = Ay + B$$

with

$$A = \frac{\beta - m}{m} \quad \text{and} \quad B = \frac{\alpha - l}{m}.$$