1.0 INTRODUCTION

Daniell P.J. obtained an extension of elementary Riemann integral to a general form of integral. He starts with a vector lattice $L$ of bounded real valued functions on a set $X$. Then a nonnegative linear functional $I$ which is continuous under monotone limits, is defined. This functional $I$ is called a Daniell Integral. Then $I$ is extended to a larger class of functions retaining all the properties of $L$ and having additional properties. As per the example cited in Loomis [LO] taking $L$ to be the class of continuous functions on $[0,1]$ and $I$ to be the ordinary Riemann integral, the extension of $L$ is then the class of Lebesgue summable functions and the extended $I$ becomes the ordinary Lebesgue integral. Taking the class $L_u$ of limits of monotone increasing sequences of functions in $L$, $I$ is extended to $L_u$ and proved that $L_u$ is a vector lattice. The fuzzy form of the above part of Daniell's procedure is discussed in this chapter. Here we define a fuzzy vector lattice and fuzzy Daniell functional. P. Lubczonok [LU], Godfrey C. Muganda [MU] and others have already given the definition of fuzzy vector

* Some of the results given in this chapter have already appeared in J. Fuzzy Maths 3(1995).
space and we make use of this definition in the forthcoming discussion.

1.1. FUZZY VECTOR LATTICES

Let $X$ be any set and $L$ be a vector lattice of extended real valued functions on $X$.

Notation 1.1.1. A fuzzy set $s$ in $L$ is a map $s : L \rightarrow [0,1]$. For $f \in L$, $\alpha \in (0,1]$, a fuzzy set $f_\alpha$ is a fuzzy point when

$$f_\alpha(h) = \alpha \quad \text{if} \quad h = f$$
$$= 0 \quad \text{if} \quad h \neq f \quad \forall h \in L.$$ 

If $s$ is a fuzzy set and $f_\alpha$ a fuzzy point, we say $f_\alpha$ is a fuzzy point of $s$ if $f_\alpha \leq s$, i.e., $\alpha \leq s(f)$.

Convention. In this thesis, a fuzzy set $s$ in $L$ is always taken to be such that $s(\varnothing) = 1$.

Definition 1.1.2 (Def. 2.1 of [LU]). A fuzzy set $s$ in $L$ is a fuzzy vector space if

$$s(\alpha f + \beta g) \geq s(f) \wedge s(g) \quad \forall f, g \in L \text{ and } \alpha, \beta \in \mathbb{R}.$$
Definition 1.1.3. \( s \) is a fuzzy vector lattice if

(i) \( s(af + bg) \geq s(f) \land s(g) \)

(ii) \( s(f \lor g) \geq s(f) \lor s(g) \)

(iii) \( s(f \land g) \geq s(f) \land s(g) \) \( \forall f, g \in L \) and \( a, b \in R \)

Notation 1.1.4. \( \tilde{s} \) denotes the set of all fuzzy points of \( s \) and \( \tilde{R} \) the set of all fuzzy points in \( R \).

Definition 1.1.5. \( f_\alpha \in \tilde{s} \) is said to be non negative and we write \( f_\alpha \geq 0 \) if \( f \geq 0 \).

Definition 1.1.6. \( \forall f_\alpha, g_\beta \in \tilde{s}, f_\alpha \leq g_\beta \) if \( f \leq g \) and \( \alpha \geq \beta \), where \( f, g \in L \) and \( \alpha, \beta \in (0,1] \). Note that \( \leq \) is a partial order in \( \tilde{s} \).

Theorem 1.1.7. \( \forall f_\alpha, g_\beta \in \tilde{s}, \)

(i) \( f_\alpha \lor g_\beta = (f \lor g)_{\min(\alpha, \beta)} \)

(ii) \( f_\alpha \land g_\beta = (f \land g)_{\max(\alpha, \beta)} \)

Proof. (i) Let \( f_\alpha, g_\beta \in \tilde{s} \). We have \( f \leq f \lor g \) and \( \alpha \geq \min(\alpha, \beta) \) and so \( f_\alpha \leq (f \lor g)_{\min(\alpha, \beta)} \); \( g \leq f \lor g \) and \( \beta \geq \min(\alpha, \beta) \) and so \( g_\beta \leq (f \lor g)_{\min(\alpha, \beta)} \). Also if \( f_\alpha, g_\beta < h_\gamma \) then \( f < h, g < h, \alpha \geq \gamma, \beta \geq \gamma \) so that...
f \lor g \leq h \text{ and } \min(\alpha, \beta) \geq \gamma. \text{ Therefore, } (f \lor g)_{\min(\alpha, \beta)} \leq h_{\gamma}.

Thus \( f_\alpha \lor g_\beta = (f \lor g)_{\min(\alpha, \beta)} \).

(ii) Follows similarly.

Remark 1.1.8. From the above theorem we find that whenever \( f_\alpha, g_\beta \in \tilde{s} \), \( f_\alpha \lor g_\beta \in \tilde{s} \) and \( f_\alpha \land g_\beta \in \tilde{s} \), i.e., \( \tilde{s} \) is having the lattice structure.

Note 1.1.9. (i) For every \( f_\alpha, g_\beta \in \tilde{s} \) and \( a, b \in \mathbb{R} \), we have

\[
a f_\alpha + b g_\beta = (a f + b g)_{\min(a^>_p)} \quad \text{from Prop. 3.1 of [MU].}
\]

Thus \( \tilde{s} \) is a vector space over \( \mathbb{R} \).

(ii) Let \( \{ (\varnothing_n)_\alpha_n \} \) and \( \{ (\gamma_m)_\beta_m \} \) be increasing sequences

\[
(\alpha_n)_{\alpha_n} \leq (\alpha_m)_{\beta_m} \quad \text{if } \lim a_n = \alpha \quad \text{and } \lim b_m = \beta > 0.
\]

(iii) \((f)_\alpha = (f^+)_{\alpha} \lor (f^-)_{\alpha} \).

Note that \( f_\alpha = (f^+)_{\alpha} \lor (f^-)_{\alpha} \).

Result 1.1.10. If \( s \) is a fuzzy vector lattice then \( \tilde{s} \) is a vector lattice.

Definition 1.1.11. A sequence \( \{ (\varnothing_n)_{\alpha_n} \} \) in \( \tilde{s} \) decreases means

\[
(\varnothing_{n+1})_{\alpha_{n+1}} \leq (\varnothing_n)_{\alpha_n} \quad \text{in } \tilde{s}, \text{ i.e., } \varnothing_{n+1} \leq \varnothing_n \quad \text{and } \alpha_{n+1} \geq \alpha_n \quad \text{in } \mathbb{R}
\]

and \( \{ (\varnothing_n)_{\alpha_n} \} \) in \( \tilde{s} \) increases means \( (\varnothing_n)_{\alpha_n} \leq (\varnothing_{n+1})_{\alpha_{n+1}} \).

\( f \) is a fuzzy vector space by Proposition 3.1 of [MU] having lattice structure and therefore it is a fuzzy vector lattice.
Definition 1.1.12. A sequence \( \{(\phi_n)_\alpha\} \) in \( s \) increases to \( \phi_\alpha (\alpha > 0) \) means \( \phi_n \uparrow \phi \) and \( \alpha_n \downarrow \alpha \).

Definition 1.1.13. \( \lim ((\phi_n)_\alpha) = \phi_\alpha (\alpha > 0) \) if \( \lim \phi_n = \phi \) and \( \lim \alpha_n = \alpha \), i.e., \( \phi_n(x) \) converges to \( \phi(x) \) for every \( x \in X \) and \( \alpha_n \to \alpha \) as \( n \to \infty \).

Remark 1.1.14. In the light of the above definition we get the following:

(i) A sequence \( \{(\phi_n)_\alpha\} \in \tilde{s} \) decreases to zero if \( \phi_n(x) \downarrow 0 \) for every \( x \in X \) and \( \alpha_n \uparrow \alpha > 0 \).

(ii) Let \( \{(\phi_n)_\alpha\} \) and \( \{((\gamma_m)_\beta\} \) be increasing sequences of fuzzy points in \( \tilde{s} \). Then \( \lim (\phi_n)_\alpha \leq \lim (\gamma_m)_\beta \)
if \( \lim \phi_n \leq \lim \gamma_m \) and \( \lim \alpha_n \geq \lim \beta_m > 0 \).

1.2. FUZZY DANIELL INTEGRAL.

\( \tilde{R} \) is a fuzzy vector space by Proposition 3.1. of [MU] having lattice structure and therefore it is a fuzzy vector lattice. Let \( \tilde{c} \) be a map from \( \tilde{s} \) to \( \tilde{R} \). Then \( \tilde{c}(f_\alpha) = \lambda_\beta \), where \( \lambda \in \mathbb{R} \), \( \beta \in (0, 1] \).

Proposition 1.2.5. If \( \tilde{c} \) is positive and linear then it is monotone. I.e., \( \tilde{c}(f_\alpha) \leq \tilde{c}(g_\alpha) \) whenever \( f_\alpha \leq g_\alpha \) and \( f_\alpha, g_\alpha \in \tilde{s} \).

Notation 1.2.1. If \( T:L \to R \) then the map from \( \tilde{s} \) to \( \tilde{R} \) defined by \( f_\alpha \to (T(f))_\alpha \) for every \( f_\alpha \in \tilde{s} \) is denoted by \( \tilde{c}_T \).
Definition 1.2.2. A map \( \tau \) from \( \mathbb{S} \) to \( \mathbb{R} \) is called linear map if \( \tau(af + bg) = a\tau(f) + b\tau(g) \) for every \( a, b \in \mathbb{R}, \) and \( f, g \in \mathbb{S} \) and for each \( f \in \mathbb{S}, \) \( \tau(f) = r \) for some \( r \in \mathbb{R}. \)

Remark 1.2.3. If \( T : \mathbb{L} \to \mathbb{R} \) is linear then \( \tau_T : \mathbb{S} \to \mathbb{R} \) is linear. For,

\[
\tau_T(af + bg) = \tau((af + bg)\min(\alpha, \beta)) = (T(af + bg))\min(\alpha, \beta)
\]

Also \( \tau_T(f) = (T(f))_\alpha. \) Hence \( \tau_T \) is linear if \( T \) is linear.

Definition 1.2.4. \( \tau : \mathbb{S} \to \mathbb{R} \) is said to be positive if \( \tau(f) > 0 \) for every \( f \in \mathbb{S}. \) i.e., \( \tau(f) = \lambda \beta \) for some \( \lambda > 0. \)

Proposition 1.2.5. If \( \tau \) is positive and linear then it is monotone. i.e., \( \tau(f) \leq \tau(g) \) whenever \( f, g \in \mathbb{S} \) and \( f \leq g. \)
Proof: Since \( f_\alpha < g_\beta \), (i.e., \( f < g \) and \( \alpha > \beta \)),
\[ g_\beta - f_\alpha = (g-f)\min(\alpha,\beta) > 0. \]
(i.e., \( \tau(g_\beta - f_\alpha) > 0 \)).
(i.e., \( \tau(g_\beta) - \tau(f_\alpha) > 0 \) since \( \tau \) is linear.
\[ i.e., s_\beta - r_\alpha > 0 \] where \( \tau(f_\alpha) = r_\alpha, \tau(g_\beta) = s_\beta \)
i.e., \( s > r \) and also we have \( \alpha > \beta \). Therefore
\[ s_\beta > r_\alpha \] i.e., \( \tau(g_\beta) > \tau(f_\alpha) \).

Note 1.2.6. Clearly, the converse of Proposition 1.2.5
is not true.

Definition 1.2.7. A linear map \( \tau: \mathcal{S} \to \mathcal{R} \) is called fuzzy
Daniell functional or fuzzy Daniell integral if for every
sequence \( \{(\varnothing_n)_{\alpha_n}\} \in \mathcal{S} \) and \( (\varnothing_n)_{\alpha_n} \downarrow 0 \), we have
\[ \lim \tau((\varnothing_n)_{\alpha_n}) = 0. \]

Remark 1.2.8. (i) If \( \tau((\varnothing_n)_{\alpha_n}) = (r_n)_{\beta_n} \), where
\( \varnothing_n \in \mathcal{L}, r_n \in \mathcal{R}, \beta_n \in (0,1] \) then \( \lim \tau((\varnothing_n)_{\alpha_n}) = 0 \)
implies \( \lim r_n = 0 \) and \( \lim \beta_n = \beta > 0 \).

(ii) If \( T: \mathcal{L} \to \mathcal{R} \) is linear then \( \lim(T(\varnothing_n)_{\alpha_n}) = 0 \)
if and only if \( \lim(T(\varnothing_n))_{\alpha_n} = 0 \) i.e., if and only if
\( \lim T(\varnothing_n) = 0 \) and \( \lim \alpha_n = \alpha > 0 \).
Lemma 1.2.9. Let \( \tau \) be a fuzzy Daniell functional. If \( \{(f_n)_\alpha\} \) and \( \{(g_m)_\beta\} \) are increasing sequences from \( \tilde{s} \) and if \( \lim (f_n)_\alpha \leq \lim (g_m)_\beta \) then

\[
\lim \tau((f_n)_\alpha) \leq \lim \tau((g_m)_\beta).
\]

Proof: Since \( \{(f_n)_\alpha\} \) and \( \{(g_m)_\beta\} \) are both increasing sequences from \( \tilde{s} \) and since \( \lim (f_n)_\alpha \leq \lim (g_m)_\beta \),

\[
(f_n)_\alpha \leq (g_m)_\beta \leq \lim (g_m)_\beta
\]

for some \( \alpha_n, \beta_m \in (0,1) \) and for sufficiently large \( n \) and \( m \). Then \( \tau((f_n)_\alpha) \leq \tau((g_m)_\beta) \) by Proposition 1.2.5. Therefore \( \lim \tau((f_n)_\alpha) \leq \lim \tau((g_m)_\beta) \).

Notation. \( \tilde{s}_u \) denotes the set of limits of all increasing sequences of fuzzy points in \( \tilde{s} \).

Lemma 1.2.10. A fuzzy Daniell functional \( \tau \) can be extended as a monotone \( \tilde{R} \) valued linear functional on \( \tilde{s}_u \), also \( \tilde{s}_u \) is a vector lattice.

Proof: Since \( \tau \) is a monotone \( \tilde{R} \) valued linear functional on \( \tilde{s} \), \( \tau(f_\alpha) \leq \tau(g_\beta) \) and \( f_\alpha \leq g_\beta \) and

\[
\tau(af_\alpha + bg_\beta) = a\tau(f_\alpha) + b\tau(g_\beta)
\]

for every \( f_\alpha, g_\beta \in \tilde{s}_u \), \( a, b \in \mathbb{R} \) by lemma 1.2.9.
Let \((\emptyset_n)_n \uparrow f \alpha \) and \((\gamma_m)_m \uparrow g \beta \), then
\[(\emptyset_n)_n \land (\gamma_m)_m \uparrow f \land g \in \tilde{s}_u \text{ and }\]
\[(\emptyset_n)_n \lor (\gamma_m)_m \uparrow f \lor g \in \tilde{s}_u .\]

For, \((\emptyset_n)_n \land (\gamma_m)_m = (\emptyset_n \land \gamma_m) \max(\alpha_n, \beta_m) \).

Since \((\emptyset_n)_n \uparrow f \alpha \) and \((\gamma_m)_m \uparrow g \beta \), we get \(\emptyset_n \uparrow f \),
\(\alpha_n \downarrow \alpha \) and \(\gamma_m \uparrow g, \beta_m \downarrow \beta \).

Therefore \((\emptyset_n \land \gamma_m) \uparrow f \land g \) and \(\max(\alpha_n, \beta_m) \downarrow \max(\alpha, \beta) \).
i.e., \((\emptyset_n)_n \land (\gamma_m)_m \uparrow f \land g \in \tilde{s}_u \). In the same way it follows that \((\emptyset_n)_n \lor (\gamma_m)_m \uparrow f \lor g \in \tilde{s}_u \).

Hence \(\tilde{s} \) is a vector lattice.

Notation 1.2.11. \(\sum_{n=1}^{\infty} (\gamma_n)_n = f \alpha \) means \(\sum_{n=1}^{\infty} \gamma_n = f \alpha \)
and \(\inf \alpha_n = \alpha \).

Lemma 1.2.12. A non negative fuzzy point \(f \alpha \) belongs to \(\tilde{s}_u \)
if and only if there is a sequence \(\{ (\gamma_n)_n \} \) of nonnegative fuzzy points in \(\tilde{s} \) such that
\[f \alpha = \sum_{n=1}^{\infty} (\gamma_n)_n \alpha_n .\]

Proof: For each \(f_n \) of nonnegative fuzzy points in \(\tilde{s}_u \),
there exists an increasing sequence \(\{ (\gamma_n)_n \} \) in \(\tilde{s} \) such that
\[f_n = \sum_{n=1}^{\infty} (\gamma_n)_n \alpha_n .\]
Proof: If \( f_\alpha \in \tilde{s}_u \) there exists an increasing sequence of nonnegative fuzzy points \( \{(\varphi_n)_\beta_n\} \) in \( \tilde{s} \) whose limit is \( f_\alpha \), and take

\[
(\varphi_1)_\alpha = (\emptyset)_\beta_1, \quad (\varphi_n)_\alpha = (\emptyset)_\beta_n - (\emptyset_{n-1})_{\beta_{n-1}} = (\emptyset_{n-1})\min(\beta_n, \beta_{n-1}) = (\emptyset_{n-1})_{\beta_{n-1}}
\]

Therefore \( \sum_{n=1}^{\infty} (\varphi_n)_\alpha_n = \lim_{n \to \infty} (\varphi_n)_\alpha_n \)

Conversely, let \( f_\alpha = \sum_{n=1}^{\infty} (\varphi_n)_\alpha_n \), we have to show that \( f_\alpha \in \tilde{s}_u \). Suppose \( (\emptyset_n)_\beta_n = \sum_{n=1}^{\infty} (\varphi_n)_\alpha_n \), where \( \varphi_n > 0 \).

Then \( (\emptyset_n)_\beta_n \in \tilde{s} \) and \( (\emptyset_n)_\beta_n \uparrow f_\alpha \). But \( \tilde{s}_u \) consists of the limit of all increasing sequences of fuzzy points in \( \tilde{s} \) so that \( f_\alpha \in \tilde{s}_u \).

Result 1.2.13. If \( \{(f_n)_\alpha_n\} \) is a sequence of nonnegative fuzzy points in \( \tilde{s}_u \) such that \( \inf \alpha_n = \alpha > 0 \), then

\[
f_\alpha = \left( \sum_{n=1}^{\infty} f_n \right) \inf \alpha_n \text{ is in } \tilde{s}_u.
\]

Also \( \tau(f_\alpha) = \sum_{n=1}^{\infty} \tau(f_n)_\alpha_n \).

Proof: For each \( (f_n)_\alpha_n \) of nonnegative fuzzy points in \( \tilde{s}_u \) there exists an increasing sequence \( \{(g_{n\xi})_{\beta_{n\xi}}\} \) in \( \tilde{s} \) such that

\[
(f_n)_\alpha_n = \left( \sum_{\xi=1}^{\infty} g_{n\xi} \right) \inf \beta_{n\xi}, \text{ where } \alpha_n = \inf \beta_{n\xi}.
\]
Therefore

\[ \sum_{n=1}^{\infty} (f_n)_{\alpha_n} = \sum_{n=1}^{\infty} \left( \sum_{l=1}^{\infty} g_{n_l} \right) \inf \beta_{n_l} \]

In the previous chapter we have seen how a fuzzy vector lattice \( \tilde{\mathcal{S}} \) is defined. Thus fuzzy Daniell integrals are defined on \( \tilde{\mathcal{S}} \) and obtained the extension of \( \tilde{\mathcal{S}} \) to \( \tilde{\mathcal{S}}' \). Here we continue our development of the fuzzy analogue of Daniell theory. In the crisp case lower and upper integrals are the same for an arbitrary function \( f \) on \( X \) with the assumption that the infimum of the empty set is \( +\infty \) are defined and in the case of integrability with respect to \( \mu \) is obtained.

The class of all \( L \)-integrable functions is denoted by \( \mathcal{L}_L \). This class \( \mathcal{L}_L \) is a vector lattice of functions containing \( L \) and also prove that \( \mathcal{L}_L \) is a positive linear functional on \( \mathcal{L}_L \). So the class \( \mathcal{L}_L \) is presented fuzzy analogue of the crisp theory as mentioned above.

2.1. UPPER FUZZY INTEGRAL AND LOWER FUZZY INTEGRAL

Definition 2.1.1. Let \( f_\alpha \) be a fuzzy point of the set of all real valued functions on \( X \). Then the upper fuzzy integral \( \mathcal{T}(f_\alpha) \) is defined by \( \mathcal{T}(f_\alpha) = \inf \mathcal{T}(g_\alpha) \). 

*Some of the results of this chapter may have been presented in Fuzzy Sets and Systems (1995).