CHAPTER 6

AN APPROACH TO MODEL SECOND-HAND PRODUCTS VIA QUASI-RENEWAL PROCESSES

6.1 INTRODUCTION

Motivated by the quasi-renewal approach in the preceding chapters which emphasized the importance of improved repairs during warranty, in contrast to a replacement, the present chapter analyzes the impact of degree of repair for second-hand products employing quasi-renewal processes. In general, the significant factors which contribute to warranty cost analysis of second-hand systems are age, usage and maintenance history. In this study the modelling is restricted to the age and/or usage of a system.

6.1.1 Notations

\[ X_i : \text{Time between (i - 1) }^{th} \text{ and i }^{th} \text{ repairs; } i = 1, 2, 3, \ldots ; \text{ independent random variables} \]

\[ A : \text{Initial age of the system} \]

\[ (X_i, Y_i) : i^{th} \text{ bivariate random variable: (age, usage); independent random variables} \]
F_i(x, y) : Cumulative distribution function of (X_i, Y_i), i = 1, 2, 3, ...
K_n(x, y) : Convolution of F_i(x, y), i = 1, 2, ..., n
C_1(W) : Expected warranty cost during W in the fixed warranty model
C_2(W) : Expected warranty cost while using a combination of parts
C_3(W) : Expected warranty cost for a system under rebate model
C(W, L) : Expected warranty cost for a system under bivariate warranty model.
E[L] : Expected length of a cycle in Model 2
C(τ, W) : Long-run average cost per unit time

6.2 MODEL FORMULATION

In response to the customers’ demand for protection against low calibre second-hand products and expensive product failures that occur shortly after purchase, warranty for second-hand products has gained prominence. Further, since the evaluation of the warranty cost depends on the quality and type of the repair action, repairs are modelled using quasi-renewal processes. Following Chattopadhyay and Murthy (2000) in this section four models are formulated with one and two-dimensional warranty using quasi-renewal processes.

6.2.1 Fixed Warranty Model

1. A second-hand system is put into operation at time t = 0.
2. Failures occur at random instants of time and are assumed to be independent.
3. On each failure, a repair action takes place with negligible repair time.
4. Inter repair times follow a quasi-renewal process, with parameter \( r \) and first operating time \( X_1 \).
5. If \( 0 < r < 1 \), it models imperfect repairs while \( r > 1 \) corresponds to an improved repair.
6. Failures are modelled as a one-dimensional point process.
7. Fixed free replacement warranty policy is adopted.

Based on the above assumptions the objective is to analyze a one-dimensional warranty model for second-hand products employing quasi-renewal processes, through the expected warranty cost for the system.

### 6.2.2 Rebate Model

Retaining the basic assumptions of the fixed warranty model it is further assumed, in this model that at most \( N \) repairs are done during warranty period \( W \). On each failure, a repair action takes place at a cost proportional to the working time of the system. The aim is to examine the expected warranty cost for the system.

### 6.2.3 Preventive Repair Model

A new system is considered at time \( t = 0 \), and a preventive repair, not as good as new, is carried out whenever the working time of the system reaches \( \tau \) and the system is still working. The sequence of working times are assumed to be independent and constitute a quasi-renewal process with parameter \( r \) and the first working time \( X_1 \). The objective is to find the
optimal preventive repair time using the long-run average cost per unit time for the system.

6.2.4 Bivariate Warranty Model

Supplementing the basic assumptions of the fixed warranty model, a free replacement bivariate warranty model is considered, in which failures are represented by a two-dimensional point process. The two-dimensional warranty is characterized by a region $\Omega$ in a two-dimensional plane with one dimension representing age and the other usage. The sequence $(X_i, Y_i)$ forms a quasi-renewal process with parameter $(u, v)$ and the bivariate random variable $(X_i, Y_i)$. If $(0 < u < 1, 0 < v < 1)$ it models imperfect repairs while $(u > 1, v > 1)$ corresponds to an improved repair. While it is mathematically convenient and may not be a realistic option to ignore the maintenance history, the factors affecting the second-hand products in this model are age and usage of the system. For a model of this kind the goal is to obtain the expected warranty cost for the system.

6.3 COST ANALYSIS

This section aims to formulate the cost equations for the various models proposed in Section 6.2.

6.3.1 Expected Warranty Cost For The Fixed Warranty Model

In this subsection, the expected warranty cost for a second-hand system is derived when sold with free replacement warranty over a fixed
warranty period $W$. It is known that for a second-hand system, the probability of the first failure is different from that of subsequent failures. Suppose that a second-hand system is of age $A = a$ then the first failure time distribution is given by

$$F_1(x) = \frac{F(x + a) - F(a)}{F(a)} \quad (6.1)$$

Since the sequence of operating times $X_n$'s are quasi-renewals, the cumulative distribution function of $X_n$ is given by

$$F_n(x) = F_1 \left( \frac{x}{r_{r+1}} \right) \frac{F \left( \frac{x}{r_{r+1}} + a \right) - F(a)}{F(a)}, \quad n = 1, 2, 3, \ldots \quad (6.2)$$

and the expected number of repairs in the fixed warranty period $W$ is given by

$$M_q(W) = E[N(W)] \sum_{n=0}^{\infty} n P\{N(W) = n\}$$

$$= \sum_{n=0}^{\infty} n \left( K_n(W) - K_{n+1}(W) \right) = \sum_{n=1}^{\infty} K_n(W) \quad (6.3)$$

Subsequently the expected warranty cost for the second-hand system is given by

$$C_1(W) = c_{rp} M_q(W) \quad (6.4)$$

where $M_q(\cdot)$ is the quasi-renewal function. Further, the adoption of a combination of products for this model is analyzed by making an assumption, that the manufacturer can perform a repair by replacing the failed parts by new/used parts. Let $p$ be the probability of replacing a failed part by new and
(1 − p) be the probability of replacing the failed part by a used part. Also \( c_n \) and \( c_u \) be the replacement costs of new and used parts respectively. Then the expected warranty cost for the system is given by,

\[
C_2(W) = pc_n M_{q_1}(W) + (1 - p)c_u M_{q_2}(W)
\]

(6.5)

where \( M_{q_1}(W) \) and \( M_{q_2}(W) \) are the quasi-renewal functions associated with the replacement of new and used parts respectively. Equations (6.4) and (6.5) are further analyzed using a numerical example in Section 6.4.

### 6.3.2 Expected Warranty Cost For The Rebate Model

In this subsection, the expected warranty cost is further examined, for the rebate model proposed in Section 6.2.2. If \( c_{sp} \) is the sale price then at the time of the \( n^{th} \) repair, \( n = 1, 2, 3, ..., N \) the rebate function is given by

\[
R_n(W) = \begin{cases} 
1 - \frac{U_n}{W} c_{sp} & \text{if } U_n \leq W \\
0 & \text{otherwise}
\end{cases}
\]

(6.6)

where \( U_n = X_1 + X_2 + \ldots + X_n, U_0 = 0; n = 1, 2, 3, ..., N \)

The expected warranty cost \( C_3(W) \) takes the form

\[
C_3(W) = \sum_{n=1}^{N} \int_{0}^{W} R_n(x) dK_n(x)
\]

\[
= c_{sp} \sum_{n=1}^{N} \int_{0}^{W} (1 - \frac{x}{W}) dK_n(x)
\]

(6.7)
6.3.2.1 Optimal replacement policy

In this section it is proposed to determine optimal $N^*$ for the rebate warranty model developed in Section 6.2.2.

Now, an explicit expression for the long-run average cost rate $B(N)$ is given by

$$B(N) = \frac{\text{Expected cost in a cycle}}{\text{Expected length of a cycle}} = \frac{C_s(W)}{L(W)}$$

$$B(N) = \frac{c_s \sum_{n=1}^{W} \left(1 - \frac{n}{W}\right) dK_n(x)}{\int_{0}^{W} x dK_N(x)}$$

$$B(N) = \frac{c_s \left( \sum_{n=1}^{N} K_n(W) - \frac{1}{W} \sum_{n=1}^{N} \int_{0}^{W} x dK_n(x) \right)}{\int_{0}^{W} x dK_N(x)}$$

Further, to minimize $B(N)$ for an optimal $N^*$, consider

$$B(N + 1) - B(N) = \frac{c_s \left\{ \int_{0}^{W} x dK_{N+1}(x) \left( \sum_{n=1}^{N+1} K_n(W) - \frac{1}{W} \sum_{n=1}^{N+1} \int_{0}^{W} x dK_n(x) \right) \right\}}{\int_{0}^{W} x dK_{N+1}(x) \int_{0}^{W} x dK_N(x)}$$

(6.9)
as the denominator of Equation (6.9) is always positive, it is clear that the sign of \( B(N + 1) - B(N) \) is the same as the sign of its numerator. Consequently

\[
B(N + 1) - B(N) \geq 0 \Rightarrow B(N + 1) \geq B(N)
\]

\[
\Rightarrow \frac{\lambda r^n}{W} \left( N - \frac{r(1-r^N)}{1-r} \right) + (1-r^N)K_{N+1}(W) \geq \frac{(1-r^N)(1-r^{N+1})}{(1-r)W} + (1-r)r^NM_q(W)
\]

(6.10)

An analytic expression for an optimal policy may be obtained through the study of Equation (6.10).

**6.3.3 Long-run Average Cost Rate For Preventive Repair Model**

In this subsection the long-run average cost per unit time is obtained for the preventive repair model. The long-run average cost per unit time is given by:

\[
C(\tau, W) = \frac{\text{Expected cost in a cycle}}{\text{Expected length in a cycle}} = \frac{C + c_p + \psi_1 + \psi_2}{E[L]}
\]

(6.11)

A cycle terminates either due to expiration of warranty or a catastrophic failure whichever occurs first. The length of the cycle is given by:

\[
L = \begin{cases} 
U_{n+1} I(U_{n+1} < W, X_1 > \tau, X_2 > \tau, \ldots, X_n > \tau, X_{n+1} < \tau) \\
\text{when catastrophic failure occurs} \\
U_{n+1} I(U_n < W < U_{n+1}, X_1 > \tau, X_2 > \tau, \ldots, X_n > \tau) \\
\text{when warranty expires}
\end{cases}
\]

(6.12)
where \( \text{I}(\cdot) \) is the indicator function. Thus the expected length of a cycle is given by:

\[
E[L] = \sum_{n=1}^{\infty} E[U_{n+1} \text{I}(U_{n+1} < W, X_1 > \tau, X_2 > \tau, \ldots, X_n > \tau, X_{n+1} < \tau)] + \sum_{n=1}^{\infty} E[U_{n+1} \text{I}(U_n < W < U_{n+1}, X_1 > \tau, X_2 > \tau, \ldots, X_n > \tau)] = \sum_{n=1}^{\infty} n\tau F_{n+1}(\tau) + \sum_{n=1}^{\infty} \int_{\tau}^{W-n\tau} t dF_{n+1}(t) + \sum_{n=1}^{\infty} n\tau \overline{F}_{n+1}(W - n\tau) + \sum_{n=1}^{\infty} \int_{W-n\tau}^{\infty} t dF_{n+1}(t)
\]

(6.13)

6.3.3.1 Expected cost due to catastrophic failure

This section provides an analytic expression for the expected cost for the system when a catastrophic failure occurs.

Let \( \chi_1 = \sum_{n=1}^{\infty} E[c_f \text{I}(U_{n+1} < W, X_1 > \tau, X_2 > \tau, \ldots, X_n > \tau, X_{n+1} < \tau)] = c_f F_{n+1}(W - n\tau) \prod_{i=1}^{n} \overline{F}_1\left(\frac{\tau}{\tau_{i+1}}\right) F_{n+1}(\tau) \)

(6.14)

6.3.3.2 Expected cost due to warranty expiration

The aim of this section is to obtain the expected cost due to the expiration of warranty.

Let \( \psi_1 = \sum_{n=1}^{\infty} c_w \text{I}(U_n < W < U_{n+1}, X_1 > \tau, X_2 > \tau, \ldots, X_n > \tau) = c_w \sum_{n=1}^{\infty} \overline{F}_{n+1}(W - n\tau) \prod_{i=1}^{n} \overline{F}_1\left(\frac{\tau}{\tau_{i+1}}\right) \)

(6.15)

6.3.3.3 Expected reward rate

Considering \( c_r \) to be the reward rate of the system, the expected reward rate of the system has been derived in this section.
Let \( \psi_4 = \sum_{n=1}^{\infty} (-c_{rr}) E \left[ U_{n+1} I (U_{n+1} < W, X_1 > \tau, X_2 > \tau, \ldots, X_n > \tau, X_{n+1} < \tau) \right] + \sum_{n=1}^{\infty} (-c_{rr}) E \left[ U_n I (U_n < W < U_{n+1}, X_1 > \tau, X_2 > \tau, \ldots, X_n > \tau) \right] \\
= -c_{rr} \sum_{n=1}^{\infty} n \tau F_{n+1} (\tau) - c_{rr} \sum_{n=1}^{\infty} \int_{\tau}^{W - n \tau} \tau dF_{n+1} (t) \\
- c_{rr} \sum_{n=1}^{\infty} n \tau F_{n+1} (W - n \tau) - c_{rr} \sum_{n=1}^{\infty} \int_{W - n \tau}^{\infty} \tau dF_{n+1} (t) \quad (6.16)

Using Equations (6.13) to (6.16) in (6.11), the objective is to obtain \( \tau^* \). Summarizing the above results the following theorem can be stated:

**Theorem 6.1:** The long-run average cost rate for preventive repair model is

\[
C(\tau,W) = \frac{C + c_p + \psi_3 + \psi_4}{E[L]}
\]

Next, the cost formulations for a second-hand system are derived when offered with a two-dimensional warranty policy.

### 6.3.4 Expected Warranty Cost For The Bivariate Warranty Model

This sub-section develops the cost equations for the model considered in Section 6.2.4. As observed before, for a second-hand system, the probability of the first failure is different from that of subsequent failures. Let us suppose that initially the system is of age \( A = a \) and usage \( U = b \) then the distribution of the time to first failure is given by

\[
F_1(x,y) = \frac{F(x+a, y+b) - F(a,b)}{F(a,b)} \quad (6.17)
\]

since \((X_i, Y_i)\) forms a quasi-renewal process, the cumulative distribution function \( F_n(x,y) \) is given by
\[ F_n(x, y) = F_1 \left( \frac{x}{n^{\alpha-1}}, \frac{y}{n^{\beta-1}} \right) \frac{F \left( \frac{x}{n^{\alpha-1}} + a, \frac{y}{n^{\beta-1}} + b \right) - F(a, b)}{F(a, b)}, \ n = 1, 2, 3, \ldots \] (6.18)

then the expected number of repairs in the warranty region \( \Omega \) is given by

\[
M_q(W, L) = E \left[ N(W, L; A, U) \right] \sum_{n=0}^{\infty} nP \left\{ N(W, L; A, U) = n \right\} 
= \sum_{n=0}^{\infty} n \left( K_n(W, L) - K_{n+1}(W, L) \right) 
= \sum_{n=1}^{\infty} K_n(W, L) 
\] (6.19)

and subsequently the expected warranty cost for the second-hand system is given by

\[
C(W, L) = c_M M_q(W, L) 
\] (6.20)

where \( M_q(\cdot, \cdot) \) is the bivariate quasi-renewal function. Using \( C(W, L) \) the cost implications of aged and used products are analyzed under a two-dimensional warranty policy. The various expected costs obtained in this section are studied by means of a numerical example.

### 6.4 NUMERICAL ILLUSTRATIONS

In this section, a numerical example is presented to illustrate the applicability of the developed models using quasi-renewal processes. The inter repair times are assumed to be exponentially distributed with means \( \lambda_n, n = 1, 2, 3, \ldots \) and probability density function \( f_n(x) = \lambda_n \exp \{-\lambda_n x\}, \ x \geq 0, \lambda_n > 0 \). Since \( X_i \)'s are quasi-renewals \( f_n(x) = \frac{\lambda_i}{r^{n-1}} \exp \left\{ -\frac{\lambda_i x}{r^{n-1}} \right\}, x \geq 0, \lambda_i > 0; \) and \( E[X_n] = \frac{r^{n-1}}{\lambda_i} \). On choosing the values of the model parameters as...
r = 1.6, \( \lambda_i = 0.06, c_{rp} = 100 \), the sensitivity of the expected warranty cost is investigated for the fixed warranty model. Figure 6.1 demonstrates that the cost of an improved repair is lesser when compared to an imperfect repair. In Figure 6.2, keeping the other parameters unchanged it is remarked that as \( \lambda_i \) increases the mean time to failure decreases. Thus, the expected warranty cost increases with an increase in the number of repairs. Further, Equation (6.5) is examined for the various cases when repairs are performed using new or used parts, as outlined below:

Case 1: \( p = 1; q = 0 \) : This is the case when the system, on failure is replaced by new parts only. In Figure 6.3, the bold line depicts the highest expected warranty cost for the system as expected.

Case 2: \( p > q \) : This corresponds to the case in which the manufacturer does a few repairs, by using new parts while the others with used parts. As, the probability of new parts is more, the expected warranty cost is higher as seen in Figure 6.3 (refer smooth line with dots).

Case 3: \( p = q \) : The smooth line in Figure 6.3, demonstrates the effect of equally likely repairs.

Case 4: \( p < q \) : In this case the probability of replacement by used parts is more when compared to new parts and hence the dotted line in Figure 6.3 illustrates lesser expected warranty cost when compared to the former three cases.

Case 5: \( p = 0; q = 1 \) : The present case occurs when a manufacturer employs used parts. The fact that the cost of repairing a second-hand system by a used part will be relatively cheaper, is reflected in the least expected warranty cost as depicted in Figure 6.3.

Hence it is concluded that the consideration of used parts while repairing a warranted second-hand product would be cost-effective.
Figure 6.1  Comparison of improved and imperfect repairs for fixed warranty model
Figure 6.2  Expected warranty cost for different $\lambda_1$
Figure 6.3  Combination of products for the fixed warranty model.
Next the fixed warranty model is analyzed through the Weibull distribution, whose density is given by \( f_n(x) = \beta \lambda_n^\beta (x)^{\beta-1} \exp \{-\lambda x\}^\beta, \ x \geq 0, \)
\( \lambda_n > 0, \beta > 0. \) For the parameters \( \lambda_i = 0.06, \beta = 1.7, c_{rp} = 100 \) from Table 6.1 it is noticed that the improved repair cost is lesser than the imperfect repair cost. Further Tables 6.2 and 6.3, respectively, exhibit the sensitivity of \( \lambda_i \) and \( \beta \) when the other parameters are fixed. From Table 6.2, when \( \lambda_i \) is increasing, the expected warranty cost is increasing and Table 6.3 shows that the expected warranty cost is decreasing when \( \beta \) is increasing. Thus \( C_1(W) \) is sensitive to small changes in the parameters of various distributions. Further when \( r = 1, \) the present model reduces to Chattopadhyay and Murthy (2000). Also, Table 6.4 explores the impact of quasi-renewal improved repairs on \( C_1(W) \) in the particular case when \( \lambda_i = 0.283, \beta = 1.25, c_{rp} = 200, \) demonstrating that the improved repair cost is lesser when compared to that of Chattopadhyay and Murthy (2000).

### Table 6.1 \( C_1(W) \) for an improved and imperfect repair

<table>
<thead>
<tr>
<th></th>
<th>Improved Repair</th>
<th>Imperfect Repair</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>( W ) A</td>
<td></td>
<td></td>
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<tr>
<td>1</td>
<td>0.1114</td>
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<td>2</td>
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<td>3</td>
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<tr>
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<td>0.5792</td>
</tr>
<tr>
<td>5</td>
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<td>0.6750</td>
</tr>
<tr>
<td>6</td>
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<td>0.7651</td>
</tr>
<tr>
<td>7</td>
<td>0.4247</td>
<td>0.8509</td>
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</table>
### Table 6.2  Sensitivity of $\lambda_i$ using $C_i(W)$

<table>
<thead>
<tr>
<th>A</th>
<th>$\lambda_i=0.06$</th>
<th>$\lambda_i=0.5$</th>
<th>$\lambda_i=1.5$</th>
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<tbody>
<tr>
<td>1</td>
<td>0.4785</td>
<td>16.2906</td>
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<td>2</td>
<td>0.7403</td>
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<td>7</td>
<td>1.7075</td>
<td>46.8974</td>
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### Table 6.3  Sensitivity of $\beta$ using $C_i(W)$

<table>
<thead>
<tr>
<th>A</th>
<th>$\beta=1.5$</th>
<th>$\beta=1.7$</th>
<th>$\beta=2.5$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1.2834</td>
<td>0.4785</td>
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<tr>
<td>2</td>
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<td>0.7403</td>
<td>0.2124</td>
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<td>1.5380</td>
<td>1.0216</td>
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<td>7</td>
<td>1.8415</td>
<td>1.7075</td>
<td>1.2788</td>
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Table 6.4  Comparison of fixed warranty model (6.2.1) with Chattopadhyay and Murthy (2000)

<table>
<thead>
<tr>
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<tbody>
<tr>
<td></td>
<td>Improved Repair</td>
<td>Imperfect Repair</td>
</tr>
<tr>
<td>W</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>3.9442</td>
<td>185.0200</td>
</tr>
<tr>
<td>1.0</td>
<td>7.9055</td>
<td>299.7122</td>
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</tr>
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<td>470.0097</td>
</tr>
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</table>

Now, the expected warranty cost for the rebate model is examined under one-dimensional warranty policy. When \( \lambda_i = 0.06 \) and \( c_{rp} = 100 \) it is to be noted from Figure 6.4 that there is an increasing pattern in \( C_3(W) \) as \( W \) increases and also improved repair cost is lesser in contrast to an imperfect repair cost. A similar examination of \( C(\tau,W) \) obtained in Theorem 6.1 when \( \lambda_i = 0.06, r = 1.6, W = 2, a = 3, c_w = 5, c_f = 3, c_{rr} = 2.4, C = 1000 \) and \( c_p = 500 \) yields the optimal preventive repair time \( \tau^* = 600 \). Further, when \( \tau^* = 600 \), a comparison of the expected cost in a cycle for the preventive repair model (numerator of \( C(\tau^* = 600,W) = -6.8995 \times 10^6 \)) with that of fixed warranty model \( (C_1(W) = 0.3479) \) illustrates that the expected cost in a cycle with optimal preventive repair time is less, depicting the importance of preventive repair for a second-hand product with warranty.
Figure 6.4  Comparison of improved and imperfect repairs for rebate warranty model
Further, the numerical results are inspected for the bivariate warranty model using bivariate Weibull distribution, whose distribution, (Refer to Baik et al. 2004) is given by,

\[ F(x, y) = \exp \left( -\left( \frac{x}{\theta_1} \right)^{\beta_1} + \left( \frac{y}{\theta_2} \right)^{\beta_2} \right), \theta_1, \theta_2, \beta_1, \beta_2 > 0 \text{ and } 0 < \delta \leq 1 \]

The values of the various parameters of the bivariate warranty model are considered to be \( \theta_1 = 5, \theta_2 = 0, \beta_1 = 4.6, \beta_2 = 2.2, \delta = 0.99, \) and \( c_{rp} = 1000. \)

Tables 6.5 and 6.6 depict the increasing pattern of \( C(W, L) \) in Equation (6.20) for different values of \( (W, L) \), as well, exhibiting a lesser cost for an improved repair when compared to an imperfect repair, as it should be.

**Table 6.5** \( C(W, L) \) for different combinations of \( (W, L) \) for an improved repair

<table>
<thead>
<tr>
<th>( L )</th>
<th>( W )</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.2432</td>
<td>0.6071</td>
<td>0.8436</td>
<td>0.9604</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.2602</td>
<td>0.6199</td>
<td>0.8565</td>
<td>0.9765</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.2775</td>
<td>0.6329</td>
<td>0.8695</td>
<td>0.9925</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.2950</td>
<td>0.6461</td>
<td>0.8826</td>
<td>1.0086</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.3129</td>
<td>0.6595</td>
<td>0.8958</td>
<td>1.0246</td>
<td></td>
</tr>
</tbody>
</table>
Table 6.6  \( C(W, L) \) for different combinations of \((W, L)\) for an imperfect repair

<table>
<thead>
<tr>
<th>L ( W )</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.0239</td>
<td>1.0523</td>
<td>1.0595</td>
<td>1.0605</td>
</tr>
<tr>
<td>0.4</td>
<td>1.0548</td>
<td>1.1044</td>
<td>1.1188</td>
<td>1.1209</td>
</tr>
<tr>
<td>0.6</td>
<td>1.0858</td>
<td>1.1560</td>
<td>1.1780</td>
<td>1.1816</td>
</tr>
<tr>
<td>0.8</td>
<td>1.1166</td>
<td>1.2070</td>
<td>1.2370</td>
<td>1.2425</td>
</tr>
<tr>
<td>1.0</td>
<td>1.1475</td>
<td>1.2575</td>
<td>1.2958</td>
<td>1.3036</td>
</tr>
</tbody>
</table>

Additionally the sensitivity of a two-dimensional warranty policy using \( C(W, L) \) is analyzed. From Figures 6.5 to 6.7 it is observed that as \( \delta, \beta_1, \beta_2 \) increase respectively the expected cost increases. Also, Figures 6.8 and 6.9 illustrate that the cost \( C(W, L) \) increases as \( \theta_1 \) and \( \theta_2 \) decrease respectively. Subsequently the contour plots are examined to assess the impact of two-dimensional warranty policy on the expected warranty cost. The curvature seen in the contour plot given in Figure 6.10 indicates that there is an effect of two-dimensional warranty policy on the expected warranty cost.

Consequently it seems important and appropriate to employ quasi-renewal processes while offering one/two dimensional warranty for second-hand products.
Figure 6.5  Surface plot for different $\delta$

Figure 6.6  Surface plot for different $\beta_1$
Figure 6.7  Surface plot for different $\beta_2$

Figure 6.8  Surface plot for different $\theta_1$
Figure 6.9  Surface plot for different $\theta_2$

Figure 6.10  Contour plot for $C(W,L)$
6.5 ESTIMATION OF PARAMETERS

For the one-dimensional warranty models developed in this chapter, it is assumed that the failure intensity function is given by

\[ \omega(t) = c_0 + c_1 M_q(t) + c_2 t \]  

(6.21)

where \( M_q(t) \) is the quasi-renewal function. Let \( t_i, i=1,2,3,\ldots \) be the time instants at which failures occur then the sequence of operating times \( X_i = t_i - t_{i-1}, i = 1,2,3,\ldots \) form a quasi-renewal process with parameter \( r \) and \( X_1 \). The aim of this section is to estimate the parameters \( c_0, c_1 \) and \( c_2 \) employing the method of maximum likelihood. As the parameters are estimated in accordance with the data available for estimation, two different cases are considered when the data available pertains to: (i) unknown failure distribution and (ii) known failure distribution.

Case (i)

The data available are the instants of failures \( t_i, 1 \leq i \leq n \) so that \( \{ X_i, i = 1,2,\ldots \} \) are a sequence of independent random variables with probability density function given by

\[ f_i(t) = \omega_i(t) e^{-\int_0^{\omega_i(s)} ds}, \; i = 1,2,\ldots,n. \]

Further, the likelihood function is given by

\[ L(t_1,t_2,t_3,\ldots,t_n) = \prod_{i=1}^{n} f_i(t_i) \]

\[ = \frac{1}{n(n-1)/2} \prod_{i=1}^{n} \omega_i \left( \frac{t_i}{r^{i-1}} \right) e^{-\sum_{i=1}^{n} \frac{t_i}{r^{i-1}}} \]

(6.22)
\[ L(t_1, t_2, t_3, \ldots, t_n; c_0, c_1, c_2) = \frac{1}{r^{n+1}} \prod_{i=1}^{n} \left[ c_0 + c_i M_q \left( \frac{t_i}{r \mu} \right) + c_2 \left( \frac{t_i}{r \mu} \right) \right] \exp \left[ - \sum_{i=1}^{n} \int_{0}^{t_i} \left( c_0 + c_i M_q (x) + c_2 (x) \right) dx \right] \] (6.23)

The estimators of \( c_0, c_1, c_2 \) are realized by solving the system of equations, obtained by differentiating Equation (6.23) with respect to \( c_i \)'s, \( i = 0, 1, 2 \) and setting them to zero, which after some simplifications lead to

\[ \sum_{i=1}^{n} \frac{1}{c_0 + c_i M_q \left( \frac{t_i}{r \mu} \right) + c_2 \left( \frac{t_i}{r \mu} \right)} = \sum_{i=1}^{n} \frac{t_i}{r^{i+1}} \] (6.24)

\[ \sum_{i=1}^{n} \frac{M_q \left( \frac{t_i}{r \mu} \right)}{c_0 + c_i M_q \left( \frac{t_i}{r \mu} \right) + c_2 \left( \frac{t_i}{r \mu} \right)} = \sum_{i=1}^{n} \int_{0}^{t_i} M_q (x) dx \] (6.25)

\[ \sum_{i=1}^{n} \frac{\left( \frac{t_i}{r \mu} \right)}{c_0 + c_i M_q \left( \frac{t_i}{r \mu} \right) + c_2 \left( \frac{t_i}{r \mu} \right)} = \frac{1}{2} \sum_{i=1}^{n} \left( \frac{t_i}{r^{i+1}} \right)^2 \] (6.26)

Case (ii)

In this case the parameters are estimated when the failure distribution is known. Suppose that a repairable system is observed until the occurrence of \( n \) failures, with the failure times as \( 0 < t_1 < t_2 < \ldots < t_n \). Now, the likelihood function is given by

\[ L(t_1, t_2, t_3, \ldots, t_n) = \prod_{i=1}^{n} f_i (t_i) = \omega_1 (t_1) e^{-\int_{0}^{t_1} \omega_1 (x) dx} \omega_2 (t_2) e^{-\int_{0}^{t_2} \omega_2 (x) dx} \ldots \omega_n (t_n) e^{-\int_{0}^{t_n} \omega_n (x) dx} \]
The maximum likelihood estimator is given by taking the logarithm of the likelihood function and setting the first partial derivative equal to zero. Further, the log-likelihood function is given by,

$$\log L = \sum_{i=1}^{n} \log \omega_i(t_i) - \sum_{i=1}^{n} \int_{0}^{t_i} \omega_i(x) \, dx$$

Since \( \{X_i, \ i = 1, 2, \ldots\} \) is a quasi-renewal process,

$$\log L = \sum_{i=1}^{n} \log \left( \frac{t_i}{r_i^{i-1}} \right) - \sum_{i=1}^{n} \int_{0}^{\frac{n}{r_i^{i-1}}} \omega_i(x) \, dx$$

where \( r \) is the quasi-renewal parameter. In the particular case when \( X_i \)'s are exponentially distributed with \( \omega_i(x) = \lambda_i, \)

$$\log L = \sum_{i=1}^{n} \log \lambda_i - \sum_{i=1}^{n} \lambda_i \frac{t_i}{r_i^{i-1}} = n \log \lambda_i - \lambda_i \sum_{i=1}^{n} \frac{t_i}{r_i^{i-1}} = 0 \Rightarrow \hat{\lambda_i} = \frac{n}{\sum_{i=1}^{n} \frac{t_i}{r_i^{i-1}}}$$

Thus, the parameter estimation carried out may help the practitioners in making flexible decisions to model repairable systems that are deteriorating or improving.

### 6.6 CONCLUSIONS

In this chapter one and two-dimensional warranty policies for second-hand products using quasi-renewal inter repair times have been
analyzed. Four cost models have been developed based on various warranty policies. In the first three models the system failures are represented by a one-dimensional point process while in the fourth model the failures are represented by a two-dimensional point process. It is readily apparent from the numerical illustrations that the stochastic models developed in this chapter emphasize the advantage of improved repairs when modelling second-hand products using quasi-renewal processes.