CHAPTER III

SOME DECOMPOSITIONS OF $\alpha$-CONTINUITY
IN TOPOLOGICAL SPACES

This chapter consists of six sections. We introduce and study the notions of $\alpha$-strong semiclosed set and $\alpha$-strong semicontinuity in topological spaces in Sections 1 and 2 respectively. In Sections 3 and 4, we study some properties of $\alpha g$-continuity and $g\alpha^{**}$-continuity in topological spaces respectively. Section 5 deals with some generalized contra continuous maps. In the last section of this chapter, we obtain two decompositions of $\alpha$-continuity, two decompositions of contra $\alpha$-continuity and two decompositions of $\alpha$-closed maps in topological spaces. The following are equivalent to the notion of $\alpha$-continuity: semicontinuity and precontinuity [78], precontinuity and $D(\alpha, p)$-continuity [86], simply continuity and precontinuity [86], semicontinuity and $D(\alpha, s)$-continuity [31] and $\beta$-continuity and $D(\alpha, ps)$-continuity [31].

3.1. $\alpha$-STRONG SEMICLOSED SETS

In this section, we introduce and study the notion of $\alpha$-strong semiclosed set in topological spaces.

**Definition 3.1.1.** A subset $S$ of $X$ is called $\alpha$-strong semiclosed if $S = A \cap F$, where $A$ is regular open and $F$ is $\alpha$-closed in $X$. 
Proposition 3.1.2. In a topological space $X$,

(a) Every regular open set is an $\alpha$-strong semiclosed set.

(b) Every $\alpha$-closed set is an $\alpha$-strong semiclosed set.

(c) Every strong semiclosed set is an $\alpha$-strong semiclosed set.

Proof. The proof is trivial.

However, the converses need not be true as seen from the following examples.

Example 3.1.3. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ The set $\{a\}$ is $\alpha$-strong semiclosed but not $\alpha$-closed. The set $\{c\}$ is $\alpha$-strong semiclosed but not regular open in $(X, \tau)$.

Example 3.1.4. Let $Y = \{x, y, z\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. The set $\{y\}$ is $\alpha$-strong semiclosed but not strong semiclosed in $(Y, \sigma)$.

Remark 3.1.5. (a) The complement of an $\alpha$-strong semiclosed set in $X$ need not be $\alpha$-strong semiclosed in $X$ and (b) finite union of $\alpha$-strong semiclosed sets in $X$ need not be $\alpha$-strong semiclosed set in $X$ as seen from the following example.

Example 3.1.6. Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c, d\}, X\}$. In $(X, \tau)$, the sets $\{a\}, \{b\}$ are $\alpha$-strong semiclosed but $\{a, b\}$ and $X - \{b\} = \{a, c, d, e\}$ are not $\alpha$-strong semiclosed.
Lemma 3.1.7. [44] The following are equivalent for a subset $S$ of $X$:

(a) $S$ is an $\alpha'$-set. That is, $\text{int}(S) = \text{int}(\text{cl}(\text{int}(S)))$,

(b) $S$ is semipreclosed.

Lemma 3.1.8. [86] Let $S$ be a subset of $X$. Then $S \in D(\alpha, p)$ if and only if $X-S \in D(\alpha, p)$.

Proposition 3.1.9. Let $S$ and $T$ be subsets of $X$.

(a) If $S$ and $T$ are $\alpha$-strong semiclosed sets in $X$, then $S \cap T$ is also $\alpha$-strong semiclosed in $X$,

(b) If $S$ is $\alpha$-strong semiclosed in $X$, then $S$ is semiclosed in $X$,

(c) If $S$ is $\alpha$-strong semiclosed in $X$, then $S$ is $\beta$-closed in $X$,

(d) If $S$ is $\alpha$-strong semiclosed in $X$, then $S$ is an $\alpha'$-set in $X$,

(e) If $S$ is $\alpha$-strong semiclosed in $X$, then $S$ is a $B$-set in $X$,

(f) If $S$ is $\alpha$-strong semiclosed in $X$, then $S \in D(\alpha, p)$, $S \in D(c, \alpha)$, $S \in D(c, p)$ and $X-S \in D(\alpha, p)$ in $X$,

(g) If $S$ is $\alpha$-strong semiclosed in $X$, then $S$ is simply open in $X$.

Proof. Proof of (a): Let $S = A \cap B$ and $T = C \cap D$, where $A$, $C$ are regular open and $B$, $D$ are $\alpha$-closed in $X$. Using Lemma 2.1.8, we obtain,

$$\text{int}(\text{cl}(A \cap C)) = A \cap C.$$ 

Hence, $S \cap T = (A \cap C) \cap (B \cap D)$. Therefore $S \cap T$ is $\alpha$-strong semi closed in $X$. 

43
Proof of (b): Let \( S = A \cap B \), where \( A \) is regular open and \( B \) is \( \alpha \)-closed in \( X \).

Using Lemma 2.1.8. and Lemma 2.1.7., we get,

\[
\text{int(cl}(S)) = \text{int(cl}(A)) \cap \text{int(cl}(B)) = A \cap \text{int}(B) = \text{int}(A) \cap \text{int}(B) = \text{int}(S).
\]

Therefore by Lemma 2.1.7., \( S \) is semiclosed.

Since every semiclosed set is \( \beta \)-closed, the proof of (c) follows.

The proof of (d) follows from Lemma 3.1.7.

Since a set is semiclosed in \( X \) if and only if it is a \( t \)-set in \( X \) and since every \( t \)-set is a \( B \)-set [100], the proof of (e) follows.

Proof of (f): Let \( S \) be \( \alpha \)-strong semiclosed in \( X \). Then by (b), \( S \) is semiclosed and by Lemma 2.1.7., \( \text{int(cl}(S)) = \text{int}(S) \). By (d) \( \text{int}(S) = \text{int(cl}(\text{int}(S))) \). Therefore \( \text{int(cl}(S)) = \text{int}(S) = \text{int(cl}(\text{int}(S))) \). This implies \( S \in D(\alpha, p) \), \( S \in D(c, p) \) and \( S \in D(c, \alpha) \) and by Lemma 3.1.8., \( X - S \in D(\alpha, p) \).

The proof of (g) follows from Lemma 2.1.6.

However, the converses need not be true as seen from the following example.

**Example 3.1.10.** Let \( X = \{a, b, c, d, e\} \) and \( \tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c, d\}, X\} \).

In \((X, \tau)\), the sets \( \{a, b, c\} \) and \( \{a, d, e\} \) are not \( \alpha \)-strong semiclosed but \( \{a, b, c\} \cap \{a, d, e\} = \{a\} \) is \( \alpha \)-strong semiclosed. Also, the set \( \{a, b, c\} \) is semiclosed (=\( t \)-set), \( \beta \)-closed (=\( \alpha^* \)-set), a \( B \)-set, a \( D(\alpha, p) \)-set, a \( D(c, p) \)-set, a \( D(c, \alpha) \)-set and simply open in \((X, \tau)\). Further, \( X - \{a, b, c\} = \{d, e\} \in D(\alpha, p) \) but not an \( \alpha \)-strong semiclosed set in \((X, \tau)\).
Proposition 3.1.11. A subset $S$ of $X$ is $\alpha$-strong semiclosed in $X$ if and only if $S = A \cap \alpha \text{cl}(S)$, where $A$ is regular open in $X$.

Proof. Assume that $S$ is $\alpha$-strong semiclosed in $X$. Then $S = A \cap F$, where $A$ is regular open and $F$ is $\alpha$-closed in $X$. This implies, $S \subseteq A$ and $S \subseteq F$. But $S \subseteq \alpha \text{cl}(S) \subseteq F$. So, $S \subseteq A \cap \alpha \text{cl}(S) \subseteq A \cap F = S$. Therefore, $S = A \cap \alpha \text{cl}(S)$.

The converse is trivial.

Remark 3.1.12. The notion of $\alpha$-strong semiclosed set is independent of the notions of $A$-set, LC-set, $D(\alpha, s)$-set, $D(\alpha, ps)$-set, preclosed set, $\alpha g$-closed set and $g\alpha^{**}$-closed set as seen from the following examples.

Example 3.1.13. Let $(X, \tau)$ be as in Example 3.1.10. In $(X, \tau)$, $\{a, d\}$ is a $D(\alpha, ps)$-set and a $D(\alpha, s)$-set but not an $\alpha$-strong semiclosed set. Also, in $(X, \tau)$, $\{a, b, c\}$ is an $A$-set and an LC-set but not $\alpha$-strong semiclosed. The set $\{b\}$ is an $\alpha$-strong semiclosed set but is neither an $A$-set nor an LC-set in $(X, \tau)$.

Example 3.1.14. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. In $(X, \tau)$, the set $\{b, c\}$ is an $\alpha$-strong semiclosed set but is neither a $D(\alpha, s)$-set nor a $D(\alpha, ps)$-set. Also, the set $\{a\}$ is $\alpha$-strong semiclosed but is neither $\alpha g$-closed nor $g\alpha^{**}$-closed in $(X, \tau)$.
Example 3.1.15. Let \((X, \tau)\) be as in Example 3.1.10. In \((X, \tau)\), the set \(\{a\}\) is \(\alpha\)-strong semiclosed but not preclosed. Let \(Y = \{a, b, c\}\) and \(\sigma = \{\emptyset, X\}\). In \((Y, \sigma)\), the set \(\{c\}\) is preclosed, \(\alpha g\)-closed and \(g\alpha^{**}\)-closed but not \(\alpha\)-strong semiclosed.

Remark 3.1.16. A \(D(c, p)\)-set is a \(D(\alpha, p)\)-set. However, the converse need not be true [31]. Now, summing up, we have the following implications. None of them is reversible.

3.2. \(\alpha\)-STRONG SEMICONTINUITY

In this section, we introduce and study the notion of \(\alpha\)-strong semicontinuity in topological spaces.

Definition 3.2.1. A map \(f: X \to Y\) is said to be \(\alpha\)-strong semicontinuous if \(f^{-1}(F)\) is \(\alpha\)-strong semiclosed in \(X\) for every \(F\) closed in \(Y\).

Theorem 3.2.2. If \(f: X \to Y\) is \(\alpha\)-continuous, then it is \(\alpha\)-strong semicontinuous.
Proof. Let \( f: X \to Y \) is \( \alpha \)-continuous and let \( F \) be a closed set in \( Y \). Then \( f^{-1}(F) \) is \( \alpha \)-closed in \( X \). By (b) of Proposition 3.1.2., \( f^{-1}(F) \) is \( \alpha \)-strong semiclosed in \( X \). Therefore \( f \) is \( \alpha \)-strong semicontinuous.

However, the converse need not be true as seen from the following example.

**Example 3.2.3.** Let \( X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}, Y = \{a, b\} \) and \( \sigma = \{\emptyset, \{a\}, Y\} \). Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = b \) and \( f(b) = f(c) = a \). Then \( f \) is \( \alpha \)-strong semicontinuous but is neither \( \alpha \)-continuous nor precontinuous nor \( D(\alpha, s) \)-continuous nor \( D(\alpha, ps) \)-continuous.

**Theorem 3.2.4.** If \( f: X \to Y \) is \( \alpha \)-strong semicontinuous, then it is semicontinuous.

Proof. Assume that \( f: X \to Y \) is \( \alpha \)-strong semicontinuous. Let \( F \) be a closed set in \( Y \). Then \( S = f^{-1}(F) \) is \( \alpha \)-strong semiclosed in \( X \). By (b) of Proposition 3.1.9., \( S \) is semiclosed in \( X \) and by Theorem 2.2.5., \( f \) is semicontinuous.

However, the converse need not be true as seen from the following example.

**Example 3.2.5.** Let \( X = \{a, b, c, d, e\}, Y = \{a, b\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c, d\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, Y\} \). Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = f(b) = f(c) = b \) and \( f(d) = f(e) = a \). Then \( f \) is \( \beta \)-continuous, semicontinuous and \( D(\alpha, p) \)-continuous but not \( \alpha \)-strong semicontinuous.

47
Corollary 3.2.6. If f: X→Y is $\alpha$-strong semicontinuous, then it is semi precontinuous.

Proof. The proof follows from (c) of Proposition 3.1.9.

However, the converse need not be true as seen from the Example 3.2.5.

Theorem 3.2.7. If f: X→Y is $\alpha$-strong semicontinuous, then f is D($\alpha$, p)-continuous.

Proof. The proof follows from (f) of Proposition 3.1.9.

The converse of the above theorem need not be true as seen from Example 3.2.5.

Lemma 3.2.8. [13] If a map f: X→Y is semi-continuous, then f is simply continuous. However, the converse need not be true.

We give two more examples and complete the section with a sum up in the form of a remark.

Example 3.2.9. Let X = {a, b, c} and $\tau = \{\phi, X\}$ and (Y, $\sigma$) be as in Example 3.2.5. Define f: (X, $\tau$) → (Y, $\sigma$) by f(a) = f(b) = a and f(c) = b. Then f is precontinuous but not $\alpha$-strong semicontinuous.

Example 3.2.10. Let X = {a, b, c}, Y = {a, b}, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Define f: (X, $\tau$) → (Y, $\sigma$) by f(a) = b and f(b) = f(c) = a. Then f is D($\alpha$, ps)-continuous and D($\alpha$, s)-continuous but not $\alpha$-strong semicontinuous.
Remark 3.2.11. From the results obtained in this section, we see that α-strong semicontinuity is independent of precontinuity, D(α, s)-continuity and D(α, ps)-continuity. Also we see that α-strong semicontinuity is strictly weaker than α-continuity whereas semicontinuity, D(α, p)-continuity, simply continuity and β-continuity are strictly weaker than α-strong semicontinuity.

3.3. αg-CONTINUITY

In this section, we prove that αg-continuity is independent of seven weaker forms of α-continuity that appear in the decompositions of α-continuity mentioned in the beginning of this chapter and we also show that α-strong semicontinuity is independent of αg-continuity.

Theorem 3.3.1. [23] Every α-continuous map f: X → Y is αg-continuous.

However, the converse need not be true as seen from the following example.

Example 3.3.2. Let X = {a, b, c} and τ = {ϕ, {a}, X}. Define f: (X, τ) → (X, τ) by f(a) = b, f(b) = c and f(c) = a. Then f is αg-continuous but is neither α-continuous nor semicontinuous nor precontinuous nor β-continuous nor α-strong semicontinuous.

Example 3.3.3. Let X = {a, b, c}, Y = {a, b}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a}, Y}. Define f: (X, τ) → (Y, σ) by f(a) = b and f(b) = f(c) = a. Then f is α-strong semicontinuous and semicontinuous but not αg-continuous.
Example 3.3.4. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Define $f: (X, \tau) \rightarrow (X, \tau)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then $f$ is precontinuous but not $\alpha g$-continuous.

Example 3.3.5. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Define $f: (X, \tau) \rightarrow (X, \tau)$ by $f(a) = b$ and $f(b) = f(c) = a$. Then $f$ is $D(\alpha, ps)$-continuous but not $\alpha g$-continuous.

Example 3.3.6. Let $X = \{a, b, c\}$, $Y = \{a, b\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = a$ and $f(b) = f(c) = b$. Then $f$ is $\alpha g$-continuous but not $D(\alpha, p)$-continuous.

Example 3.3.7. Let $X = \{a, b, x, y, z\}$, $Y = \{a, b\}$, $\tau = \{\emptyset, \{a, b\}\}$, $\sigma = \{\emptyset, \{a\}, Y\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = f(x) = f(z) = a$ and $f(b) = f(y) = b$. Then $f$ is $\alpha g$-continuous but not $D(\alpha, s)$-continuous.

Example 3.3.8. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Define $f: (X, \tau) \rightarrow (X, \tau)$ by $f(a) = b$ and $f(b) = f(c) = a$. Then $f$ is simply continuous but not $\alpha g$-continuous.

Example 3.3.9. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Define $f: (X, \tau) \rightarrow (X, \tau)$ by $f(a) = f(b) = a$ and $f(c) = c$. Then $f$ is $\alpha g$-continuous but not simply continuous.
Remark 3.3.10. From [86] and [31] we have the following implications in the form of diagrams:

\[
\begin{align*}
\alpha\text{-continuity} & \quad \Rightarrow \quad \text{semi continuity} & \quad \Rightarrow \quad \beta\text{-continuity} \\
\alpha\text{-continuity} & \quad \Rightarrow \quad \text{pre continuity} & \quad \Rightarrow \quad \beta\text{-continuity}
\end{align*}
\]

However none of the implications can be reversed. From the above implications and from Examples 3.3.2 - 3.3.9, we realize that $\alpha g$-continuity is independent of semicontinuity, pre continuity, $\beta$-continuity, $D(\alpha, ps)$-continuity, $D(\alpha, p)$-continuity, $D(\alpha, s)$-continuity, simple continuity and $\alpha$-strong semicontinuity.

3.4. $g\alpha^{**}$- CONTINUITY

In this section, we study some properties of $g\alpha^{**}$-continuous maps in topological spaces. First, we recall the following two definitions.
Definition 3.4.1. [57] A subset $S$ of $X$ is said to be $g\alpha^{**}$-closed in $X$ if $\text{acl}(S) \subseteq \text{int}(\text{cl}(U))$, whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$. A subset $S$ of $X$ is called $g\alpha^{**}$-open in $X$ if its complement is $g\alpha^{**}$-closed in $X$.

Definition 3.4.2. [57] A map $f: X \to Y$ is said to be $g\alpha^{**}$-continuous if $f'(F)$ is $g\alpha^{**}$-closed in $X$ for each $F$ closed in $Y$.

Lemma 3.4.3. The following are equivalent for a subset $S$ of a space $X$.

(a) $g\alpha^{**}$-closed,

(b) $\text{acl}(S) \subseteq U$, whenever $A \subseteq U$ and $U$ is regular open in $X$.

Proof. (a) $\to$ (b): Let $S$ be $g\alpha^{**}$-closed and let $S \subseteq U$ where $U$ is regular open in $X$. Since every regular open set is $\alpha$-open, $U$ is $\alpha$-open in $X$. Then by the definition of $g\alpha^{**}$-closed sets, we have $\text{acl}(S) \subseteq \text{int}(\text{cl}(U))$. But $\text{int}(\text{cl}(U)) = U$, because $U$ is regular open. Therefore $\text{acl}(S) \subseteq U$. Hence, (a) $\to$ (b).

(b) $\to$ (a): Assume that $S$ satisfies (b) and let $S \subseteq U$ where $U$ is $\alpha$-open in $X$. Since $U$ is $\alpha$-open, $U \subseteq \text{int}(\text{cl}(\text{int}(U)))$

\[ \subseteq \text{int}(\text{cl}(U)). \]

So, we have $S \subseteq U \subseteq \text{int}(\text{cl}(U))$. That is, $S \subseteq \text{int}(\text{cl}(U))$. Since $\text{int}(\text{cl}(U))$ is regular open, by (b), we obtain $\text{acl}(S) \subseteq \text{int}(\text{cl}(U))$. Hence (b) $\to$ (a).

Remark 3.4.4. $\text{acl}(X-A) = X - \alpha\text{int}(A)$, for any subset $A$ of $X$. 

52
Theorem 3.4.5. A subset $S$ of $X$ is $g\alpha^{**}$-open if and only if $\alpha\text{int}(S) \supseteq F$ whenever $S \supseteq F$ and $F$ is regular closed in $X$.

Proof. Necessity: Let $S$ be $g\alpha^{**}$-open and let $S \supseteq F$ where $F$ is regular closed in $X$. Then $X-S \subseteq X-F$, where $X-F$ is regular open in $X$. Since, $S$ is $g\alpha^{**}$-open, $X - S$ is $g\alpha^{**}$-closed in $X$. Now, using Lemma 3.4.3., we see that $\alpha\text{cl}(X-S) \subseteq X - F$. By Remark 3.4.4., $X - \alpha\text{int}(S) \subseteq X - F$. That is, $\alpha\text{int}(S) \supseteq F$.

Sufficiency: Assume that $\alpha\text{int}(S) \supseteq F$ whenever $S \supseteq F$ and $F$ is regular closed in $X$. Let $X - S \subseteq U$, where $U$ is regular open in $X$. Then $S \supseteq X - U$, where $X - U$ is regular closed in $X$. By hypothesis, $\alpha\text{int}(S) \supseteq X - U$. That is, $X - \alpha\text{int}(S) \subseteq U$. That is, $\alpha\text{cl}(X - S) \subseteq U$. So, $X - S$ is $g\alpha^{**}$-closed in $X$, by Lemma 3.4.3. Therefore, $S$ is $g\alpha^{**}$-open in $X$.

Theorem 3.4.6. A map $f: X \to Y$ is $g\alpha^{**}$-continuous if and only if $f^{-1}(U)$ is $g\alpha^{**}$-open in $X$ for each open set $U$ in $Y$.

Proof. The proof is trivial.

Proposition 3.4.7. (a) Every $\alpha\text{g}$-closed set in $X$ is $g\alpha^{**}$-closed in $X$ and (b) If $f: X \to Y$ is $\alpha\text{g}$-continuous, then $f$ is $g\alpha^{**}$-continuous.
Proof. Let $S$ be $\alpha g$-closed in $X$. Let $S \subseteq U$ where $U$ is regular open in $X$. Since every regular open set in $X$ is open in $X$, $U$ is open in $X$. Since $S$ is $\alpha g$-closed in $X$, $\alpha\text{cl}(S) \subseteq U$. Hence by Lemma 3.4.3, $S$ is $g\alpha^{**}$-closed in $X$.

Now, let $f: X \rightarrow Y$ be $\alpha g$-continuous and let $F$ be a closed set in $Y$. Then $S = f^{-1}(F)$ is $\alpha g$-closed in $X$. By (a), $S$ is $g\alpha^{**}$-closed in $X$. Therefore, $f$ is $g\alpha^{**}$-continuous.

However, the converses need not be true as seen from the following example.

Example 3.4.8. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. In $(X, \tau)$, the set $\{a, b\}$ is $g\alpha^{**}$-closed but not $\alpha g$-closed. Also, let $Y = \{x, y\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = f(b) = y$ and $f(c) = x$. Then $f$ is $g\alpha^{**}$-continuous but not $\alpha g$-continuous.

Theorem 3.4.9. ([57], [23]) If a map $f: X \rightarrow Y$ is $\alpha$-continuous, then it is $g\alpha^{**}$-continuous. However, the converse need not be true.

Theorem 3.4.10. Let $f: X \rightarrow Y$ be a $D(\alpha, s)$-continuous map. Then $f$ is $g\alpha^{**}$-continuous.

Proof. Let $U$ be an open set in $Y$. Then $S = f^{-1}(U)$ is a $D(\alpha, s)$ set in $X$. So, $\alpha\text{int}(S) = \text{sint}(S)$. Let $S \supseteq F$, where $F$ is regular closed in $X$. This implies, $\text{cl}(\text{int}(S)) \supseteq \text{cl}(\text{int}(F)) = F$. Hence $S \cap \text{cl}(\text{int}(S)) \supseteq F$. That is, $\text{sint}(S) \supseteq F$. That is, $\alpha\text{int}(S) \supseteq F$. Therefore $S$ is $g\alpha^{**}$-open in $X$ and the proof follows.
However the converse need not be true as seen from the following example.

Example 3.4.11. In Example 3.3.7, f is $g\alpha^*$-continuous but not $D(\alpha, s)$-continuous.

Corollary 3.4.12. If $f : X \to Y$ is $D(\alpha, ps)$-continuous, then $f$ is $g\alpha^*$-continuous.

Proof. Since every $D(\alpha, ps)$-set is $D(\alpha, s)$-set [31], the proof follows from Theorem 3.4.10.

However, the converse need not be true as seen from the following example.

Example 3.4.13. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{b, c\}, X\}$. Define $f : (X, \tau) \to (X, \tau)$ by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then $f$ is $g\alpha^*$-continuous but is neither $D(\alpha, ps)$-continuous nor $D(\alpha, p)$-continuous.

Now, we give three more examples, follow them with some remarks before concluding this section.

Example 3.4.14. Let $X = \{a, b, c\}$, $Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$ and $f(b) = f(c) = a$. Then $f$ is $D(\alpha, p)$-continuous and semicontinuous but not $g\alpha^*$-continuous.

Example 3.4.15. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$. Define $f : (X, \tau) \to (X, \tau)$ by $f(a) = b$ and $f(b) = f(c) = a$. Then $f$ is $g\alpha^*$-continuous but is neither $\beta$-continuous nor semicontinuous nor precontinuous.
Example 3.4.16. Let $X = \{a, b, c, d\}$, $Y = \{a, b\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Define $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = f(b) = f(d) = a$ and $f(c) = b$. Then $f$ is precontinuous and $\beta$-continuous but not $g\alpha^{**}$-continuous.

Remark 3.4.17. From [86], [31] and from the results obtained in this section, we have the following diagrams:

\[
\begin{array}{c}
\text{D}(\alpha, p)\text{-continuity} \\
\text{D}(\alpha, ps)\text{-continuity} \\
\text{D}(\alpha, s)\text{-continuity}
\end{array}
\]

\[
\begin{array}{c}
\text{g}\alpha^{**}\text{-continuity} \\
\alpha g\text{-continuity}
\end{array}
\]

Remark 3.4.18. The map defined in Example 3.3.8. is simply continuous but not $g\alpha^{**}$-continuous. The map in Example 3.3.9. is $g\alpha^{**}$-continuous but not simply continuous. The map defined in Example 3.3.2. is $g\alpha^{**}$-continuous but not $\alpha$-
strong semicontinuous. The map defined in Example 3.3.3. is $\alpha$-strong semicontinuous but not $g\alpha^{**}$-continuous. Thus $g\alpha^{**}$-continuity is independent of $\alpha$-strong semicontinuity and simply continuity.

3.5. SOME GENERALIZED CONTRA CONTINUOUS MAPS IN TOPOLOGICAL SPACES

In this section, we introduce the notions of contra $\alpha$-continuity, contra $\alpha$-strong semicontinuity, contra $\alpha$g-continuity and contra $g\alpha^{**}$-continuity and study some relations between them.

**Definition 3.5.1.** A map $f: X \to Y$ said to be contra $\alpha$-continuous if $f^{-1}(V)$ is $\alpha$-closed in $X$ for each open set $V$ in $Y$.

**Definition 3.5.2.** A map $f: X \to Y$ said to be contra $\alpha$-strong semicontinuous if $f^{-1}(V)$ is $\alpha$-strong semiclosed in $X$ for each open set $V$ in $Y$.

**Definition 3.5.3.** A map $f: X \to Y$ said to be contra $\alpha$g-continuous if $f^{-1}(V)$ is $\alpha$g-closed in $X$ for each open set $V$ in $Y$.

**Definition 3.5.4.** A map $f: X \to Y$ said to be contra $g\alpha^{**}$-continuous if $f^{-1}(V)$ is $g\alpha^{**}$-closed in $X$ for each open set $V$ in $Y$.

**Remark 3.5.5.** Contra $\alpha$-continuity and $\alpha$-continuity are independent as seen from the following examples.
Example 3.5.6. Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, X\} \). Define \( f: (X, \tau) \to (X, \tau) \) by \( f(a) = b, f(b) = a \) and \( f(c) = c \). Then \( f \) is contra \( \alpha \)-continuous but is neither contra continuous nor \( \alpha \)-continuous.

Example 3.5.7. Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, X\} \). Define \( f: (X, \tau) \to (X, \tau) \) by \( f(a) = f(b) = a \) and \( f(c) = b \). Then \( f \) is \( \alpha \)-continuous but not contra \( \alpha \)-continuous.

Remark 3.5.8. contra \( \alpha \)-strong semicontinuity is independent of \( \alpha \)-strong semicontinuity as seen from the following example.

Example 3.5.9. The map defined in Example 3.5.6. is contra \( \alpha \)-strong semicontinuous but not \( \alpha \)-strong semicontinuous. The map defined in Example 3.5.7. is \( \alpha \)-strong semicontinuous but not contra \( \alpha \)-strong semicontinuous.

Remark 3.5.10. contra \( \alpha g \)-continuity is independent of \( \alpha g \)-continuity as seen from the following examples.

Example 3.5.11. Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{a, b\}, X\} \).

Define \( f: (X, \tau) \to (X, \tau) \) by \( f(a) = b, f(b) = a \) and \( f(c) = c \). Then \( f \) is \( \alpha g \)-continuous but not contra \( \alpha g \)-continuous.

Example 3.5.12. Let \( X = \{a, b, c, d, e, f\} \), \( \tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{a, b, c, d\}, X\} \), \( Y = \{x, y\} \) and \( \sigma = \{\phi, \{x\}, Y\} \). Define \( g: (X, \tau) \to (Y, \sigma) \) by
g(a) = g(b) = g(c) = y and g(d) = g(e) = g(f) = x. Then g is contra $\alpha$g-continuous but is neither $\alpha$g-continuous nor contra $\alpha$-continuous.

**Remark 3.5.13.** contra $g\alpha^{**}$-continuity and $g\alpha^{**}$-continuity are independent as seen from the following example.

**Example 3.5.14.** Let $(X, \tau)$ and $(Y, \sigma)$ be as in Example 3.5.12.

Define $g: (X, \tau) \to (Y, \sigma)$ by $g(a) = g(b) = x$ and $g(c) = g(d) = g(e) = g(f) = y$. Then $g$ is $g\alpha^{**}$-continuous but not contra $g\alpha^{**}$-continuous. Also, define $h: (X, \tau) \to (Y, \sigma)$ by $h(a) = h(b) = y$ and $h(c) = h(d) = h(e) = h(f) = x$. Then $h$ is contra $g\alpha^{**}$-continuous but not $g\alpha^{**}$-continuous.

**Remark 3.5.15.** contra $\alpha$-strong semicontinuity is independent of contra $\alpha$g-continuity and contra $g\alpha^{**}$-continuity as seen from the following examples.

**Example 3.5.16.** Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Define $f: (X, \tau) \to (X, \tau)$ by $f(a) = f(c) = a$ and $f(b) = b$. Then $f$ is contra $\alpha$g-continuous and contra $g\alpha^{**}$-continuous but not contra $\alpha$-strong semicontinuous.

**Example 3.5.17.** Let $X = \{a, b, c\}$, $Y = \{a, b\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma$ be the discrete topology on $Y$. Define $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = a$ and $f(b) = f(c) = b$. Then $f$ is contra $\alpha$-strong semicontinuous but is neither contra $\alpha$g-continuous nor contra $g\alpha^{**}$-continuous.
Proposition 3.5.18. Let $f: X \rightarrow Y$ be a map.

(a) If $f$ is contra continuous, then $f$ is contra $\alpha$-continuous,

(b) If $f$ is contra $\alpha$-continuous, then $f$ is contra $\alpha g$-continuous,

(c) If $f$ is contra $\alpha$-continuous, then $f$ is contra $\mathcal{g} \alpha^{**}$-continuous,

(d) If $f$ is contra $\alpha$-continuous, then $f$ is contra $\alpha$-strong semicontinuous,

(e) If $f$ is contra $\alpha g$-continuous, then $f$ is contra $\mathcal{g} \alpha^{**}$-continuous,

(f) If $f$ is contra strong semicontinuous, then $f$ is contra $\alpha$-strong semicontinuous.

Proof. The proof is trivial.

However the converses need not be true as seen from the following examples.

Example 3.5.19. The map defined in Example 3.5.6. is contra $\alpha$-continuous but not contra continuous.

Example 3.5.20. Let $X = \{a, b, c\}$, $Y = \{x, y, z, p\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{x, y\}, \{z\}, \{x, y, z\}, Y\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = x$, $f(b) = z$ and $f(c) = y$. Then $f$ is contra $\alpha g$-continuous and contra $\mathcal{g} \alpha^{**}$-continuous but not contra $\alpha$-continuous.

Example 3.5.21. Let $X = \{a, b, c\}$, $Y = \{x, y\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{x\}, X\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = x$ and $f(b) = f(c) = y$. Then $f$ is contra $\alpha$-strong semicontinuous but not contra $\alpha$-continuous.
Example 3.5.22. Let \((X, \tau)\) be as in Example 3.5.12. Let \(Y = \{x, y, z\}\) and \(\sigma = \{\emptyset, \{x\}, \{x, y\}, Y\}\). Define \(g: (X, \tau) \to (Y, \sigma)\) by \(g(a) = g(b) = g(c) = g(d) = x\), \(g(e) = y\) and \(g(f) = z\). Then \(g\) is contra \(g\)-continuous but not contra \(\alpha\)-continuous.

Example 3.5.23. Let \(X = \{a, b, c, d, e\}\), \(\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c, d\}, X\}\), \(Y = \{x, y\}\) and \(\sigma = \{\emptyset, \{x\}, Y\}\). Define \(g: (X, \tau) \to (Y, \sigma)\) by \(g(b) = g(c) = x\) and \(g(a) = g(d) = g(e) = y\). Then \(g\) is contra \(\alpha\)-strong semicontinuous but not contra strong semicontinuous.

Remark 3.5.24. Contra \(\alpha\)-continuity and contra strong semicontinuity are independent as seen from the following examples.

Example 3.5.25. Let \(X = \{a, b, c\}\), \(Y = \{x, y\}\), \(\tau = \{\emptyset, \{a\}, X\}\) and \(\sigma = \{\emptyset, \{x\}, Y\}\). Define \(f: (X, \tau) \to (Y, \sigma)\) by \(f(b) = x\) and \(f(a) = f(c) = y\). Then \(f\) is contra \(\alpha\)-continuous but not contra strong semi continuous.

Example 3.5.26. Let \(X = \{a, b, c\}\), \(Y = \{x, y\}\), \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\) and \(\sigma = \{\emptyset, \{x\}, Y\}\). Define \(f: (X, \tau) \to (Y, \sigma)\) by \(f(a) = x\) and \(f(b) = f(c) = y\). Then \(f\) is contra strong semicontinuous but not contra \(\alpha\)-continuous.
Summing up the results we have the following implications. None of them is reversible.

contra continuity \rightarrow contra \alpha-continuity \rightarrow contra \alpha g-continuity \rightarrow contra g\alpha^{**}-continuity

contra strong semicontinuity \rightarrow contra \alpha-strong semicontinuity

3.6. DECOMPOSITIONS

In this section, we obtain two decompositions of \alpha-continuity, two decompositions of \alpha-closed maps and two decompositions of contra \alpha-continuity. Also, we discuss, briefly, Asit Kumar Sen and Bhattacharyya’s [7] decomposition of \alpha-continuity.

Theorem 3.6.1. The following are equivalent for a subset S of X.

(a) S is \alpha-closed in X,
(b) S is \alpha g-closed and \alpha-strong semiclosed in X,
(c) S is g\alpha^{**}-closed and \alpha-strong semiclosed in X.

Proof. (a) \Rightarrow (b) \Rightarrow (c) are trivial.

Assume (c). Then S = A \cap F, where A is regular open and F is \alpha-closed in X. This implies S \subseteq A and S \subseteq F. Since S is g\alpha^{**}-closed, using Lemma 3.4.3, we have \alpha cl(S) \subseteq A. Since F is \alpha-closed, we have \alpha cl(S) \subseteq F. So, \alpha cl(S) \subseteq A \cap F = S. That is, \alpha cl(S) = S. Therefore (c) \Rightarrow (a).
Corollary 3.6.2. A mapping \( f: X \to Y \) is

(a) \( \alpha \)-continuous if and only if it is both \( \alpha g \)-continuous and \( \alpha \)-strong semicontinuous,

(b) \( \alpha \)-continuous if and only if it is both \( g\alpha^{**} \)-continuous and \( \alpha \)-strong semicontinuous.

Proof. The proof follows from Theorem 3.6.1.

Remark 3.6.3. From the various results and examples given in the previous sections, it is clear that there two decompositions are different from the earlier ones, stated in the beginning of the chapter. Also note that the second decomposition is an improvement over the first.

In [7], Asitkumar Sen and Bhattacharyya obtained a decomposition of \( \alpha \)-continuity: A map \( f: X \to Y \) is \( \alpha \)-continuous if and only if it is weakly \( \alpha \)-continuous and weak* \( \alpha \)-continuous.

Definition 3.6.4. A map \( f: X \to Y \) is said to be

(a) weakly \( \alpha \)-continuous [79] if for each \( x \in X \) and each open set \( V \) containing \( f(x) \) in \( Y \), there exists an \( \alpha \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq \text{cl}(V) \),

(b) weak* \( \alpha \)-continuous [7] if for each open set \( V \) in \( Y \) \( f^{-1}(\text{Fr}(V)) \) is \( \alpha \)-closed in \( X \) where \( \text{Fr}(V) \) denotes the frontier of \( V \).
Example 3.6.5. Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, X\} \), \( Y = \{a, b, c\} \) and \( \sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, Y\} \). Define \( f : X \rightarrow Y \) by \( f(a) = c \) and \( f(b) = b \). Then \( f \) is weakly \( \alpha \)-continuous, \( \alpha g \)-continuous, \( g\alpha^{**} \)-continuous but is neither \( \alpha \)-strong semicontinuous nor \( \alpha \)-continuous nor weak* \( \alpha \)-continuous.

Example 3.6.6. Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\} \), \( Y = \{a, b\} \) and \( \sigma = \{\emptyset, \{a\}, Y\} \). Define \( f : X \rightarrow Y \) by \( f(a) = b \) and \( f(b) = f(c) = a \). Then \( f \) is \( \alpha \)-strong semicontinuous, weakly \( \alpha \)-continuous but is neither \( \alpha \)-continuous nor \( \alpha g \)-continuous nor \( g\alpha^{**} \)-continuous nor weak* \( \alpha \)-continuous.

Example 3.6.7. Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, X\} \), \( Y = \{a, b, c\} \) and \( \sigma = \{\emptyset, \{a\}, \{b, c\}, Y\} \). Define \( f : X \rightarrow Y \) by \( f(a) = c \), \( f(b) = a \) and \( f(c) = b \). Then \( f \) is weak* \( \alpha \)-continuous, \( g\alpha^{**} \)-continuous and \( \alpha g \)-continuous but is neither \( \alpha \)-continuous nor \( \alpha \)-strong semicontinuous nor weakly \( \alpha \)-continuous.

Example 3.6.8. Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c\}, \{a, c, d\}, X\} \), \( Y = \{x, y, z\} \) and \( \sigma = \{\emptyset, \{x, y\}, \{z\}, Y\} \). Define \( f : X \rightarrow Y \) by \( f(a) = f(b) = f(c) = x \) and \( f(d) = z \). Then \( f \) is \( \alpha \)-strong semicontinuous and weak* \( \alpha \)-continuous but is neither \( \alpha \)-continuous nor weakly \( \alpha \)-continuous nor \( g\alpha^{**} \)-continuous nor \( \alpha g \)-continuous.
Remark 3.6.9. We have the following relationships:

(a) $\alpha$-strong semicontinuity and weakly $\alpha$-continuity are independent (Example 3.6.5 and 3.6.8).

(b) $\alpha$-strong semicontinuity and weak$^*$ $\alpha$-continuity are independent (Example 3.6.6 and 3.6.7).

(c) $\alpha g$-continuity and weakly $\alpha$-continuity are independent. (Example 3.6.6 and 3.6.7).

(d) $\alpha g$-continuity and weak$^*$ $\alpha$-continuity are independent. (Example 3.6.5 and 3.6.8).

(e) $g\alpha^{**}$-continuity and weakly $\alpha$-continuity are independent. (Example 3.6.6 and 3.6.7).

(f) $g\alpha^{**}$-continuity and weak$^*$ $\alpha$-continuity are independent. (Example 3.6.5 and 3.6.8).

(g) $\alpha$-strong semicontinuity and weakly $\alpha$-continuity need not imply $\alpha$-continuity. (Example 3.6.6).

(h) $\alpha$-strong semicontinuity and weak$^*$ $\alpha$-continuity need not imply $\alpha$-continuity. (Example 3.6.8).

(i) $\alpha g$-continuity and weakly $\alpha$-continuity need not imply $\alpha$-continuity. (Example 3.6.5).

(j) $\alpha g$-continuity and weak$^*$ $\alpha$-continuity need not imply $\alpha$-continuity. (Example 3.6.7).
(k) $g\alpha^{**}$-continuity and weakly $\alpha$-continuity need not imply $\alpha$-continuity.

(Example 3.6.5).

(l) $g\alpha^{**}$-continuity and weak* $\alpha$-continuity need not imply $\alpha$-continuity.

(Example 3.6.7).

From the above remark, it is evident that our decompositions of $\alpha$-continuity are independent of Asit kumar Sen and Bhattacharyya's.

**Definition 3.6.10.** A map $f : X \to Y$ is said to be $\alpha$-strong semiclosed if $f(F)$ is $\alpha$-strong semiclosed in $Y$ for each closed set $F$ in $X$.

**Remark 3.6.11.** A subset $S$ of $X$ is said to be $\mathcal{rg}$-closed if $\text{acl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is regular open in $X$ [81]. By Lemma 3.4.3, the notion of $\mathcal{rg}$-closed set coincide with the notion of $g\alpha^{**}$-closed set.

**Definition 3.6.12.[81]** A map $f : X \to Y$ is said to be $\mathcal{rg}$-closed if $f(F)$ is $\mathcal{rg}$-closed in $Y$ for each closed set $F$ in $X$.

**Corollary 3.6.13** A map $f : X \to Y$ is

(a) $\alpha$-closed if and only if it is both $\alpha\mathcal{g}$-closed and $\alpha$-strong semiclosed,

(b) $\alpha$-closed if and only if it is both $\mathcal{rg}\mathcal{g}$-closed and $\alpha$-strong semiclosed.

**Proof.** The proof follows from Theorem 3.6.1.
**Corollary 3.6.14.** A map $f: X \rightarrow Y$ is

(a) contra $\alpha$-continuous if and only if it is both contra $\alpha g$-continuous and contra $\alpha$-strong semicontinuous,

(b) contra $\alpha$-continuous if and only if it is both contra $g\alpha^{**}$-continuous and contra $\alpha$-strong semicontinuous.

**Proof.** The proof follows from Theorem 3.6.1.

* * * * *