Chapter 3

Dominator Colorings and Safe Clique Partitions on Graphs

Abstract

Let $G = (V, E)$ be a graph of order $p$. A dominator coloring on $G$ is a proper coloring of $G$ with the additional property that every vertex $v$ that dominates an entire color class. The smallest number of colors for which there exists a dominator coloring of $G$ is called the dominator chromatic number of $G$ and is denoted by $\chi_d(G)$. A safe clique partition of a graph $G$ is a partition of $V(G)$ where no two vertices in the same block are adjacent. The smallest number of blocks for which there exists a safe clique partition of $G$ is called the safe clique partition number of $G$ and is denoted by $\lambda(G)$.

In this chapter, we obtain bounds for $\chi_d(G)$ for bipartite graphs with minimum degree one in terms of domination number. We obtain a sufficient condition for the existence of a graph $G$ with $\Delta(G) \leq p-2$ to have no safe clique partition and the relation between the safe clique partition and the dominator coloring number. We also establish a characterization of those which have the safe clique partition and the dominator coloring number for a graph $G$. We also provide some results concerning the dominating chromatic number.
Chapter-3

Dominator Colorings and Safe Clique Partitions on Graphs

Abstract

Let $G = (V, E)$ be a graph of order $p$. A dominator coloring on $G$ is a proper coloring of $G$ with the additional property that every vertex in $G$ dominates an entire color class. The smallest number of colors for which there exists a dominator coloring of $G$ is called the dominator chromatic number of $G$ and is denoted by $\chi_d(G)$. A safe clique partition of a graph $G$ is a partition of the vertices of $G$ into complete subgraphs with the additional property that for each vertex $v$, there exists a complete subgraph that has no vertex in the open neighborhood of $v$. The smallest number of partitions for which there exists a safe clique partition of $G$ is called the safe clique partition number of $G$ and is denoted by $\chi_s(G)$. In this chapter, we obtain a bound for $\chi_d(G)$ for bipartite graphs with minimum degree one in terms of domination number. We obtain a sufficient condition for the existence of a graph $G$ with $\Delta(G) \leq p - 2$ to have no safe clique partition and the relation between the safe clique partition concerning mostly metric invariants such as diameter and radius of $G$. Also we establish a characterization of trees which have no safe clique partition and the dominator chromatic number for a caterpillar. We also obtain some results concerning the dominator chromatic number.
Main Results

Section 3.1.

In this section we discuss a bound for $\chi_d(G)$ for bipartite graphs with minimum degree one in terms of domination number.

**Theorem 3.1.** Let $G$ be a bipartite graph with $\delta(G) = 1$. Then $\gamma(G) + 1 \leq \chi_d(G) \leq \gamma(G) + 2$.

**Proof.** By Theorem (1.59), we have $\max \{\gamma(G), \chi(G)\} \leq \chi_d(G) \leq \gamma(G) + \chi(G)$.

Since $G$ is a bipartite graph with $\delta(G) = 1$, $\chi(G) = 2$. Hence $\gamma(G) \leq \chi_d(G) \leq \gamma(G) + 2$.

But, if $\chi_d(G) = \gamma(G)$, $G$ can be colored with $\gamma(G)$ colors.

Let $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_r\}$ be the collection of dominator color classes of $G$ for such a coloring.

By definition of dominator coloring, every transversal of the collection of color classes will form a $\gamma$ - set.

Let $u$ be an end vertex and $v$ be its support. Then $u$ and $v$ belong to different color classes.

This implies that exists a $\gamma$ - set contains both an end vertex and its support, which is not possible.

$\therefore \gamma(G) + 1 \leq \chi_d(G) \leq \gamma(G) + 2$.

**Remark 3.2.** We conclude with the similar proof for the theorem 1.57 shows that $\chi_d(G) \geq \gamma(G) + 1$ if $\delta(G) = 1$. 38
Corollary 3.3. $\chi_d(P_{3k}) = k + 2$, for every $k \geq 2$.

Proof. By theorem 3.1, $\chi_d(P_{3k}) = \gamma(P_{3k}) + 1$ (or) $\gamma(P_{3k}) + 2$.

$\therefore \chi_d(P_{3k}) = k + 1$ (or) $k + 2$.

Let $u_1, u_2, u_3, \ldots, u_{3k}$ be the vertices of $P_{3k}$ in order.

Let if possible, $\chi_d(P_{3k}) = k + 1$.

Then, let $\mathcal{C} = \{c_1, c_2, \ldots, c_{k+1}\}$ be a dominator color class of $P_{3k}$ for some $\chi_d$-coloring of $P_{3k}$.

Then it is clear that either $u_1$ or $u_2$ is a non-repeated color.

Suppose $u_1$ and $u_2$ are given colors 1 and 2. It is clear that $u_3$ or $u_4$ must have a color different from 1 and 2.

It follows that three colors must be represented in the first four vertices $u_1, u_2, u_3$ and $u_4$.

Let $\{u_{i1}, u_{i2}, \ldots, u_{ik+1}\}$ be a transversal of the collection $\mathcal{C}$ with $u_{ij} \in c_j$.

Then $u_{i1}, u_{i2}, u_{i3}$ belong to the dominator color classes $c_1, c_2$ and $c_3$ respectively. But three colors must be represented in the first four vertices $u_1, u_2, u_3$ and $u_4$.

We choose $\{u_{i1}, u_{i2}, u_3\} \subset \{u_1, u_2, u_3, u_4\}$.

By our dominator coloring, $\bigcup_{j=1}^{k+1} N[u_j] = V(G)$.

$\therefore \exists k = |V(G)| = \bigcup_{j=1}^{k+1} N[u_j]$.
Chapter 3: Dominator Colorings and Safe Clique Partitions on Graphs

\[ \leq \left| \bigcup_{j=1}^{3} N[u_j] \right| + \left| \bigcup_{j=4}^{k+1} N[u_j] \right| \]

\[ = 5 + 3(k + 1 - 3) \]

\[ = 3k - 1, \text{ a contradiction} \]

Thus \( \chi_d(P_{3k}) = k + 2 \), for every \( k \geq 2 \). \( \blacksquare \)

Corollary 3.4. \( \chi_d(P_{3k+i}) = k + 3 \) for \( k > 2 \), \( i = 1, 2 \) and \( k = 2, i = 2 \).

Proof. First we have to prove that \( \chi_d(P_{3k+2}) = k + 3 \) for \( k \geq 2 \).

Let \( u_1, u_2, u_3, \ldots, u_{3k+1}, u_{3k+2} \) be the vertices of \( P_{3k+2} \) in order.

By theorem (3.1), we have \( k + 2 \leq \chi_d(P_{3k+2}) \leq k + 3 \).

Let if possible \( \chi_d(P_{3k+2}) = k + 2 \).

Then, let \( C = \{c_1, c_2, \ldots, c_{k+2}\} \) be the collection of dominator color classes of \( P_{3k+2} \).

If three colors are represented in the first three vertices, then proceeding as in corollary 3.3, we have, \( 3k + 2 \leq 4 + 3(k + 2 - 3) = 3k + 1 \), a contradiction.

Thus \( \chi_d(P_{3k+2}) = k + 3 \).

Suppose that the first three vertices have only two colors.

In this case, \( u_1, u_2 \) have the same color and \( u_3 \) must have a non-repeated color.

Then we can argue by induction on \( k \).

When \( k = 2 \), we can easily verify the result.

We assume that the result is true for \( k - 1 \).

Then the coloring on the sub graph \( \langle u_4, u_5, \ldots, u_{3k+2} \rangle \) is a dominator coloring.
Chapter 3: Dominator Colorings and Safe Clique Partitions on Graphs

\[ \chi_d(P_{3k+2}) \geq 1 + \chi_d(P_{3k-1}) \]

\[ = 1 + (k - 1 + 3) \]

\[ = k + 3 \text{ for } k \geq 2. \]

Thus \( \chi_d(P_{3k+2}) = k + 3 \), for \( k \geq 2 \).

By similar argument to \( P_{3k+1} \), we can prove that, \( \chi_d(P_{3k+1}) = k + 2 \) for \( k > 2 \).

Section 3.2.

Dominator colorings on Join of two Graphs.

In this section, we prove that the dominator colorings for join of two graphs are same as its proper coloring.

Theorem 3.5. For any connected graph \( G = G_1 + G_2 \), \( \chi_d(G) = \chi(G) = \chi(G_1) + \chi(G_2) \).

Proof. Let \( G = G_1 + G_2 \).

We know that, \( \chi(G) = \chi(G_1) + \chi(G_2) \).

Also any proper coloring of \( G \) is a dominator coloring of \( G \).

\[ \therefore \chi_d(G) = \chi(G) = \chi(G_1) + \chi(G_2). \]

The following cases are particular case of the above theorem.

1. For a wheel graph \( W_p = K_1 + C_{p-1}, \chi_d(W_p) = \begin{cases} 4 & \text{if } p \text{ is even} \\ 3 & \text{if } p \text{ is odd} \end{cases} \]

2. If a graph has a vertex of full degree \( v \), \( \chi_d(G) = \chi(G) = 1 + \chi(G - v) \).

3. If \( G \) is a complete \( r \)-partite graph \( K_{m_1, m_2, \ldots, m_r} \),

\[ \chi_d(K_{m_1, m_2, \ldots, m_r}) = \chi(K_{m_1}) + \chi(K_{m_2}) + \ldots + \chi(K_{m_r}) = r. \]

41
Section 3.3.

Dominator chromatic number in caterpillars

In this section, we find the dominator chromatic number for a caterpillar of class 2. In theorem 1.60, the dominator chromatic number for a caterpillar of class 1 is found. Now we give a bound for dominator chromatic number for a caterpillar of class 2.

**Theorem 3.6.** Let $G$ be a caterpillar of class 2 having exactly $r$ vertices of degree at least 3 and $r_i$ be the number of zero strings of length $i$, $2 \leq i \leq m$, $m = \text{maximum length of a zero string in } G$. We assume $r_2 > 0$. Then

$$\chi_d(G) \leq r + 2 + \sum_{i=3}^{m} r_i \left( \left\lfloor \frac{i-2}{3} \right\rfloor \right)$$

**Proof.** Let $G$ be a caterpillar of class 2 and $r$ be the number of vertices of degree at least 3.

We assume that $r_2 > 0$.

In any dominator coloring of $G$, all vertices of degree at least 3 receive a non-repeated color, say 1 to $r$ and the vertices of each zero string of length 2 receive the two repeated colors, say $(r+1)$ and $(r+2)$ respectively. All pendant vertices in $G$ receive a repeated color, say either $(r+1)$ or $(r+2)$.

Suppose that $r_i > 0$, for some $i \geq 3$.

We note that, each zero string of length $i$, $i \geq 3$ is isomorphic to $P_i$, $i \geq 3$. 

42
Let $u_{j_1}, u_{j_2}, \ldots, u_{j_i}$ $(1 \leq j \leq r_i)$ be the vertices of $P_i$ $(3 \leq i \leq m)$ in order.

Consider any zero string of length 3.

Then the end vertices of $P_3$ has a neighbor of degree at least 3. Therefore the end vertices of $P_3$ receive color either $(r+1)$ or $(r+2)$ and the central vertex receives a non-repeated color.

Consider any zero string of length 4.

Then the end vertices $\{u_{i_1}, u_{i_4}\}$ receive color $(r+1)$, one support vertex $u_{i_2}$ receives the 2nd repeated color $(r+2)$ and other support vertex $u_{i_3}$ receives a non-repeated color respectively.

Consider any zero string of length $i$, $i \geq 5$

We note that, in any dominator coloring of a path $P_i$, $i \geq 5$, the both end vertices $\{u_{i_1}, u_{i_2}\}$ and the support vertices $\{u_{i_2}, u_{i_{i-1}}\}$ of $P_i$ receive a repeated color and non-repeated colors respectively.

Now the end vertices $\{u_{i_1}, u_{i_2}\}$ of each zero string of length $i$, $i \geq 3$ has a neighbor of degree at least 3.

Therefore the end vertices of $P_i$, $i \geq 5$ dominates the vertices of degree at least 3.

So the end vertices $\{u_{i_1}, u_{i_2}\}$ receive the repeated color $(r+1)$ (say). The support vertices $\{u_{i_2}, u_{i_{i-1}}\}$ of $P_i$, $i \geq 5$ dominates the vertices $u_{i_3}$ and $u_{i_{i-2}}$ respectively. So the support vertices $\{u_{i_2}, u_{i_{i-1}}\}$ of $P_i$, $i \geq 5$ receive another repeated color.
Chapter 3: Dominator Colorings and Safe Clique Partitions on Graphs

(r+2)(say). The vertices $u_{13}$ and $u_{14}$ receive a non-repeated color (two new colors) and the vertices $u_{14}$ and $u_{15}$ receive the already used repeated colors. Also the repeated colors of each zero string of length $i$, $i \geq 2$ are common repeated colors $(r+1)$ and $(r+2)$ respectively.

Hence $\chi_d(G) \leq r + 2 + \sum_{i=3}^{\infty} r \left\lfloor \frac{i-2}{3} \right\rfloor$.

Illustration 3.7.

\[\chi_d(G_1) = 8 = r + 2 + r_3 + r_5 \left\lfloor \frac{5-2}{3} \right\rfloor\]

\[\chi_d(G_2) = 12 = r + 2 + r_3 + r_5 \left\lfloor \frac{5-2}{3} \right\rfloor + r_7 \left\lfloor \frac{7-2}{3} \right\rfloor\]

Figure 3.1
Section 3.4.

In this section we obtain a sufficient condition for the existence of a graph G with \( \Delta(G) \leq p - 2 \) to have no safe clique partition and the relation between the safe clique partition concerning mostly metric invariants such as diameter and radius of G. Also we establish a characterization of trees which have no safe clique partition.

**Sufficient condition for the existence of a graph G with \( \Delta(G) \leq p - 2 \) to have no safe clique partition**

**Theorem 3.8.** Suppose \( u \in V(G) \) is unsafe with respect to a clique partition of G. Then \( e(u) \geq 2 \).

**Proof.** Let \( u \) be an unsafe vertex with respect to a clique partition \( \{V_1, V_2, \ldots, V_r\} \).

Let \( v \in V(G) \).

Suppose that \( uv \notin E(G) \). Let \( v \in V \) (say).

Since \( u \) is unsafe, \( N(u) \cap V_i \neq \emptyset \).

Therefore there exists \( w \in N(u) \cap V_i \Rightarrow uwv \) is a path in G

\[ \Rightarrow d(u, v) \leq 2 \]

\[ \Rightarrow e(u) \leq 2. \]

The relation between the safe clique partition and the metric invariants of G, are brought out in the following two corollaries to Theorem 3.8.
Corollary 3.9. Let $G$ be any graph of radius at least 3. Then any clique partition of $G$ is safe. In particular, any clique partition of a graph of diameter at least 5 is safe.

Corollary 3.10. Let $G$ be a graph such that either $\text{rad}(G^c)$ is at least 3 or $G^c$ is disconnected. Then $\chi_d(G) = \chi(G)$.

Proof. First suppose that $G^c$ is a connected graph of radius at least 3.

Since any proper coloring of $G$ is a clique partition of $G^c$ and $\text{rad}(G^c) \geq 3$, every vertex of $G^c$ is safe with respect to any clique partition of $G^c$.

Thus every coloring of $G$, gives rise to a safe clique partition of $G^c$.

So every proper coloring of $G$ is a dominator coloring of $G$.

Hence $\chi_d(G) = \chi(G)$.

Suppose $G^c$ is disconnected.

Then $G = G_1 + G_2$ for sub graphs $G_1$ and $G_2$ of $G$.

It is then clear that any color partition of $G$ is also a dominator color partition of $G$. Therefore $\chi_d(G) = \chi(G) = \chi(G_1) + \chi(G_2)$.

The following result is proved in [8]. We give an alternate proof.

Theorem 3.11. Let $G$ be a tree. A maximum matching $M$ of $G$ is unsafe if and only if $G$ is one of the following.

46
(i) $G$ is a star $K_{1,r}$.

(ii) $G$ is a bistar $B(1,r)$ where $r \geq 1$.

(iii) $G$ is a graph obtained by subdividing at most $(r - 1)$ edges of a star $K_{1,r}$, where $r$ is the degree of a central vertex in $G$.

**Proof.**

Let $G$ be a tree and $M$ be a maximum matching in $G$.

Take a clique partition $U$ arising from $M$ such that a vertex $u$ is unsafe with respect to $U$.

Then by Theorem 3.8, $e(u) \leq 2$ and $\text{diam}(G) \leq 4$.

If $\text{rad}(G) = 1$ and $\text{diam}(G) \leq 2$, $G$ is a star $K_{1,r}$ and any clique partition of $G$ is unsafe.

Suppose that $\text{rad}(G) = 2$ and $\text{diam}(G) = 3$.

We note that a tree of radius 2 and diameter 3 is a bistar. That is a tree with only two non pendant vertices ($p \geq 4$) [These two vertices form the centre of the star].

If one of these vertices is of degree 2, then the other central vertex is unsafe.

Thus the bistar which are unsafe are those with a vertex of degree 2. Such graphs are also wounded spiders (When exactly one edge of the star $K_{1,r}$ ($r \geq 1$) is subdivided).

Hence $G$ is a bistar $B(1,r)$, where $r \geq 1$.

Suppose that $\text{rad}(G) = 2$ and $\text{diam}(G) = 4$.

Then $G$ has a unique centre $u$. 
Let $\deg(u) = r$ and $N(u) = \{u_1, u_2, \ldots, u_r\}$ and $n_i$ be the number of pendant edges incident on $u_i$, $1 \leq i \leq r$.

We may assume that $n_1 \geq n_2 \geq \ldots \geq n_r$.

If $n_i > 1$, for some $i$, then any clique partition of $G$ is safe. Therefore we assume $n_i \leq 1$, $\forall i = 1, 2, \ldots, r$.

We consider the following two cases.

**Case 1:** $n_i = 0$, $\forall i = 1, 2, \ldots, r$.

Then $\text{diam}(G) = 2$, a contradiction.

Therefore $\exists m$ ($1 \leq m \leq r$) such that $n_m \neq 0$.

**Case 2:** $n_i = 1$, $\forall i = 1, 2, \ldots, r$.

Then any clique partition of $G$ is safe.

Therefore $\exists n$ ($1 \leq n \leq r$) such that $n_n \neq 1$.

---

**Figure 3.2**

A wounded spider
By cases (1) and (2), \( \exists k, k < r \) such that \( n_s = 1, \ 1 \leq s \leq k \) and \( n_j = 0, \ k + 1 \leq j \leq r. \)

So \( u \) is unsafe with respect to a clique partition \( U. \)

Thus \( G \) is a graph obtained by subdividing at most \((r-1)\) edges of a star \( K_{1,r}, \)

where, \( r \) is the degree of a central vertex in \( G. \)

The converse is obvious.◼