Chapter 2

Preliminaries

In this chapter, we introduce the notion of distances which are useful in the subsequent chapters. For more information, we refer toHarary [10].

Definition 1.1

A graph \( G = (V, E) \) is a finite set \( V \) of vertices, together with a set \( E \) of edges, a pair of vertices \( u, v \) is called adjacent if \( u, v \in V \) and there is an edge \( uv \in E \). A graph \( G \) is connected if for any pair \( u, v \) of \( G \) there is a path from \( u \) to \( v \).

Definition 1.2

The degree of a vertex \( v \) in \( G \) is the number of edges \( v \) is incident to, denoted \( \deg(v) \). A simple graph \( G \) contains an isolated vertex if a vertex of degree 0 is contained in \( G \). A graph \( G \) has no isolated vertices if each vertex of \( G \) is adjacent to a non-empty number of other vertices of \( G \).
Chapter - 1

Preliminaries

Abstract

In this chapter, we discuss the basic definitions and some theorems which are useful in the subsequent chapters. For graph theoretic terminology we refer to Hararry [10].

Definition 1.1.

A graph \( G = (V, E) \) is a finite non-empty set \( V = V(G) \) of \( p \) points (vertices) together with a prescribed set \( E = E(G) \) of \( q \) unordered pairs of distinct points of \( V \). Each pair \( e = \{u, v\} \) of points is a line (edge) of \( G \) and \( e \) is said to be join \( u \) and \( v \). We write \( e = uv \) and say that \( u \) and \( v \) are adjacent points (vertices). If two distinct lines (edges) are incident with a common point, then they are adjacent lines (edges). A graph with \( p \) points and \( q \) lines is called a \((p, q)\)-graph. The \((1,0)\)-graph is trivial.

Definition 1.2.

The degree of a vertex \( v \) in a graph \( G \) is the number of edges of \( G \) incident with \( v \) and is denoted by \( \text{deg} (v) \). A vertex of degree 0 is called an isolated vertex and a vertex of degree 1 is called an end vertex or a pendant vertex of \( G \). Any vertex which is adjacent to a pendant vertex is called a support.
Theorem 1.3. [10] For a \((p, q)\)-graph, \(\sum_{i=1}^{p} \deg(v_i) = 2q\).

Remark 1.4.

If \(G\) is a \((p, q)\) - graph, then \(0 \leq \deg(v) \leq p - 1\), for any vertex \(v\). The minimum and maximum degrees of vertices of \(G\) are denoted by \(\delta(G)\) and \(\Delta(G)\) respectively.

Definition 1.5.

A graph \(G\) is regular of degree \(r\) if every vertex of \(G\) has degree \(r\). Such graphs are called \(r\)-regular graphs. A 3-regular graph is called a cubic graph.

Definition 1.6.

A graph is complete if every two of its vertices are adjacent. A complete \((p, q)\)-graph is therefore a regular graph of degree \(p - 1\) having \(q = \frac{p(p - 1)}{2}\) edges. We denote this graph by \(K_p\).

Definition 1.7.

A sub graph \(H\) of a graph \(G\) is a graph having all its points and lines in \(G\). A spanning sub graph of \(G\) is a sub graph containing all the points of \(G\). For any set \(S\) of points of \(G\), the induced sub graph \(\langle S \rangle\) is the maximal sub graph of \(G\) with point set \(S\).

Definition 1.8.

A clique of a graph \(G\) is a maximal complete sub graph of \(G\). The clique number of \(G\) is the order of the maximal complete sub graph and is denoted by \(\omega(G)\).
Definition 1.9.
A graph without any edges is called a null graph or an empty graph.

Definition 1.10.
Two graphs $G_1$ and $G_2$ are isomorphic if there exists a one-to-one correspondence between their point sets which preserves adjacency. That is, there exists a bijection $\Phi : V(G_1) \rightarrow V(G_2)$ such that $uv \in E(G_1) \iff \Phi(u)\Phi(v) \in E(G_2)$.

Definition 1.11.
The complement $G^c$ or $\overline{G}$ of a graph $G$ also has $V(G)$ as vertex set, but two vertices are adjacent in $G^c$ if and only if they are not adjacent in $G$. A self-complementary graph is a graph which is isomorphic with its complement.

Definition 1.12.
A bipartite graph $G$ is a graph whose vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every line of $G$ joins $V_1$ with $V_2$. If $G$ contains every line joining $V_1$ and $V_2$, then $G$ is a complete bipartite graph. If $|V_1| = m, |V_2| = n$ then the complete bipartite graph is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is called a star. The vertex of degree $n$ is called its centre. The graph obtained by joining the centers of two stars $K_{1,r}$ and $K_{1,s}$ by an edge is defined to be a bistar and is denoted by $B(r, s)$.
Definition 1.13.

A complete multipartite graph $K_{n_1, n_2, \ldots, n_m}$ is the graph with vertex set $V = V_1 \cup V_2 \cup \ldots \cup V_m$, where $|V_i| = n_i$ for $1 \leq i \leq m$, $\{u, v\} \subseteq V_i$ implies $u$ and $v$ are non-adjacent and $u \in V_i$ and $v \in V_j$ with $i \neq j$ implies $u$ and $v$ are adjacent.

Definition 1.14.

Let $u$ and $v$ be (not necessarily distinct) vertices of a graph $G$. A $u$-$v$ walk of $G$ is a finite alternating sequence $u = u_0e_1u_1e_2\ldots u_{n-1}e_nu_n = v$ of vertices and edges beginning with vertex $u$ and ending with vertex $v$ such that $e_i = u_{i-1}u_i$ for $i = 1, 2, \ldots, n$. The number $n$ is called the length of a walk. A walk $u_0e_1u_1e_2\ldots u_{n-1}e_nu_n$ is determined by the sequence $u_0u_1\ldots u_{n-1}u_n$ of its vertices and hence we specify a walk simply by $(u_0u_1\ldots u_{n-1}u_n)$. A walk in which all the vertices are distinct is called a path. A walk $(u_0u_1\ldots u_{n-1}u_n)$ is called a closed walk if $u_0 = u_n$. A closed walk in which $u_0, u_1, u_2, \ldots, u_{n-1}$ are distinct is called a cycle. A path on $p$ vertices is denoted by $P_p$ and a cycle on $p$ vertices is denoted by $C_p$.

Definition 1.15.

A graph $G$ is said to be connected if any two vertices of $G$ are joined by a path. A maximal connected sub graph of $G$ is called a component of $G$. Thus a disconnected graph has at least two components.
Definition 1.16.

The distance \(d(u,v)\) between two vertices \(u\) and \(v\) is the smallest number of edges in a path between \(u\) and \(v\) in \(G\) if it exists. The eccentricity, \(e(u)\), of a vertex \(u\) is the largest distance from \(u\) to any vertex of \(G\). The radius of \(G\) is \(\min\{e(u) : u \in V(G)\}\) and the diameter of \(G\) is \(\max\{e(u) : u \in V(G)\}\).

Definition 1.17.

A graph \(G\) is called acyclic if it has no cycles. A connected acyclic graph is called a tree.

Definition 1.18.

A caterpillar is a tree with the additional property that the removal of all pendant vertices leaves a path. This path is called the spine of the caterpillar, and the vertices of the spine are called vertebrae. A vertebra which is not a support is called a zero string. In a caterpillar, any sequence of exactly \(i\)-consecutive zero strings is called a zero string of length \(i\). A caterpillar which has no zero string of length at least 2 is said to be of class 1 and all other caterpillars are of class 2.

Definition 1.19.

The join of \(n\) vertex disjoint graphs \(G_1, G_2, \ldots, G_n\) with vertex sets \(V_1, V_2, \ldots, V_n\) respectively is denoted by \(G_1 + G_2 + \ldots + G_n\) and is defined by \(G_1 \cup G_2 \cup \ldots \cup G_n\) and all lines joining \(V_i\) with \(V_j \ \forall i \neq j \text{ and } 1 \leq i, j \leq n\).
Definition 1.20.
A wheel is a graph obtained from a cycle by adding a new vertex and edges it to all the vertices of a cycle. A wheel with \( p \) vertices is denoted by \( W_p = K_1 + C_{p-1} \).

Definition 1.21.
An edge \( e = uv \) of a graph \( G \) is said to be subdivided if \( e \) is replaced by the edges \( uw \) and \( wv \) for some vertex \( w \notin V(G) \).

Definition 1.22.
A wounded spider is the graph obtained by subdividing at most \( r-1 \) of the edges of a star \( K_{1,r} \) for \( r \geq 2 \).

Definition 1.23[15].
The corona of two graphs \( G_1 \) and \( G_2 \) is defined to be the graph \( G = G_1 \circ G_2 \) formed from one copy of \( G_1 \) and \( |V(G_1)| \) copies of \( G_2 \) where the \( i^{th} \) vertex of \( G_1 \) is adjacent to every vertex in the \( i^{th} \) copy of \( G_2 \).

Definition 1.24.
A subset \( S \) of \( V \) in a graph \( G \) is said to be independent if no two vertices in \( S \) are adjacent. The maximum number of vertices in an independent set is called the independence number of \( G \) and is denoted by \( \beta(G) \). A set \( F \) of edges in a graph \( G \) are said to be independent if no two of the edges in \( F \) are adjacent. A matching of \( G \) is a set of independent edges in \( G \). The maximum cardinality of an independent
set of edges in $G$ is called the edge independence number of $G$ and is denoted by $\beta_1(G)$. A matching $M$ is maximum if there is no matching $M'$ with $|M'| > |M|$.

**Definition 1.25.**

The size of smallest maximal independent set of $G$ is called independent domination number of $G$ and is denoted by $i(G)$.

**Definition 1.26.**

A vertex and an edge are said to cover each other if they are incident. A set of vertices which covers all the edges of a graph $G$ is called a cover for $G$. A set of edges which covers all the vertices of a graph $G$ is called an edge cover for $G$. The smallest number of vertices in any cover for $G$ is called its covering number of $G$ and is denoted by $\alpha(G)$. The smallest number of edges in any edge cover for $G$ is called the edge covering number of $G$ and is denoted by $\alpha_1(G)$. An edge cover exists if and only if $\delta(G) > 0$.

**Definition 1.27.**

A set of independent edges covering all the vertices of a graph $G$ is called a 1-factor or a perfect matching of $G$.

**Definition 1.28.**

A graph $G$ is decomposable if $G$ can be expressed as a join of two proper subgraphs (equivalently $G^c$ is disconnected). A graph which is not decomposable is said to be indecomposable. If $G$ can be decomposed into $r$ proper vertex disjoint
Chapter 1: Preliminaries

A subset $S$ of $V$ is called a dominating set if every vertex in $V \setminus S$ is adjacent to some vertex in $S$.

The domination number $\gamma$ is the minimum cardinality of a dominating set of $G$. A $\gamma$-set is any dominating set.

Definition 1.29.

A proper coloring of $G$ is an assignment of colors to the vertices of $G$, such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of $G$ is called chromatic number of $G$ and is denoted by $\chi(G)$.

Let $V = \{u_1, u_2, ..., u_p\}$ and $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ be a collection of subsets $C_i \subseteq V$. Then a subset $A = \{u_1, u_2, ..., u_m\}$ is a transversal of $\mathcal{C}$ if $|A \cap C_i| = 1$, $1 \leq i \leq m$. A color represented in a vertex $u$ is called a non-repeated color if there exists one color class $C_i \in \mathcal{C}$ such that $C_i = \{u\}$.

Remark 1.30. For any non-trivial tree $T$, $\chi(T) = 2$.

Definition 1.31.

The open neighborhood $N(v)$ of a vertex $v$ is the set of all vertices adjacent to $v$. $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of $v$. The open neighborhood set $N(S)$ of a set $S$ of vertices is the set of vertices adjacent to some vertex in $S$. $N[S] = N(S) \cup S$ is called the closed neighborhood set of $S$.

Definition 1.32.
A subset $S$ of $V$ is called a dominating set if every vertex in $V - S$ is adjacent to some vertex in $S$. A dominating set $S$ is a minimal dominating set if no proper subset of $S$ is a dominating set of $G$. The domination number $\gamma$ is the minimum cardinality taken over all minimal dominating sets of $G$. A $\gamma$-set is any dominating set with cardinality $\gamma$. The upper domination number $\Gamma$ is the maximum cardinality taken over all minimal dominating sets of $G$.

**Notation 1.33.**

For any real number $x$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$ and $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.

**Theorem 1.34[15].** For any path $P$, $\gamma(P) = \left\lfloor \frac{p + 1}{3} \right\rfloor$.

**Definition 1.35.**

Let $S$ be a set of vertices of a graph $G$ and let $u \in S$. We say that a vertex $v$ is a private neighbor of $u$ (with respect to $S$) if $N[v] \cap S = \{u\}$. The private neighbor set of $u$ with respect to $S$ is defined as $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. Notice that $u \in pn[u, S]$ if and only if $u$ is an isolate in $\langle S \rangle$, in which case we say that $u$ is its own private neighbor.

**Definition 1.36.**
The private neighbor set of a set $S$ is defined as $pn(S) = \{v : pn[v,S] \neq \emptyset\}$. The private neighbor count of a set $S$ is defined to be the cardinality of $pn(S)$ and is denoted by $pnc(S)$.

The concept of irredundance number was introduced by Cockayne, Hedetniemi and Miller [5].

**Definition 1.37.**

A subset $S \subseteq V$ is irredundant if for every $v \in S$, $pn[v,S] \neq \emptyset$, i.e., every vertex $v \in S$ has at least one private neighbor. An irredundant set $S$ is called a maximal irredundant set if no proper super set of $S$ is irredundant. The minimum and maximum cardinality of a maximal irredundant set in $G$ are called the irredundance number and upper irredundance number of $G$ and are denoted by $ir(G)$ and $IR(G)$ respectively.

**Theorem 1.38**[15].

A dominating set $S$ is a minimal dominating set if and only if it is dominating and irredundant.

**Theorem 1.39** [6].

For any graph $G$, $IR(G) \leq p - \delta(G)$.

**Corollary 1.40**[7].

For any graph $G$, $\Gamma(G) \leq p - \delta(G)$ and $\beta_0(G) \leq p - \delta(G)$.

**Proposition 1.41**[6].
For any \( p \)-vertex graph \( G \) with minimum degree \( \delta \), \( IR(G) \leq p - \delta(G) \), where equality holds if and only if \( G \) is one of the following graphs:

(i) \( V(G) = X \cup W \), where \( |X| = p - \delta \), \( X \) is independent in \( G \) and each vertex in \( X \) is joined to each vertex in \( W \). The vertices in \( W \) are joined to one another arbitrarily, subject to \( \text{deg} \ w \geq \delta \) for each \( w \in W \).

(ii) \( V(G) = X \cup Y \cup Z \), where \( |X| = |Y| = p - \delta \) (i.e., \( \delta \geq \frac{p}{2} \)), \( \langle X \rangle \cong \langle Y \rangle \cong K_{p - \delta} \) and the vertices in \( X \) are joined to the vertices in \( Y \) by a matching. Further, \( |Z| = 2\delta - p \), each vertex in \( Z \) is joined to each vertex in \( X \cup Y \) and the vertices in \( Z \) are joined to one another arbitrarily, subject to \( \text{deg} \ z \geq \delta \) for each \( z \in Z \).

**Corollary 1.42** [6].

(a) If \( IR(G) = p - \delta(G) \), then \( \Gamma(G) = IR(G) \).

(b) If \( IR(G) = p - \delta(G) \), where \( \delta < \frac{p}{2} \), then \( \beta_0(G) = \Gamma(G) = IR(G) \).

**Theorem 1.43** [7].

For any graph \( G \), \( IR(G) = p - \delta(G) \) if and only if \( \Gamma(G) = p - \delta(G) \).

Cockayne et al [5] were first observed the following inequality chain.

**Theorem 1.44** [5].

For any graph \( G \), \( \text{ir}(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G) \).
The concept of total (open) domination was introduced by Cockayne, Dawes and Hedetniemi [4].

**Definition 1.45.**

A subset $S$ of $V$ is called a total dominating set if every vertex in $V$ is adjacent to some vertex in $S$. A total dominating set $S$ is a minimal total dominating set if no proper subset of $S$ is a total dominating set of $G$. The total domination number $\gamma_t$ is the minimum cardinality taken over all minimal total dominating sets of $G$. A $\gamma_t$-set is any minimal total dominating set with cardinality $\gamma_t$. Clearly, total dominating sets exist in $G$ if and only if $\delta(G) > 0$.

**Theorem 1.46** [12].

For $p \geq 3$, $\gamma_t(P_p) = \gamma_t(C_p) = \left\lceil \frac{p}{2} \right\rceil$ if $p \equiv 0, 1, 3 \pmod{4}$

$\left\lceil \frac{p}{2} \right\rceil + 1$ if $p \equiv 2 \pmod{4}$

Furthermore if $p \equiv 2, 3 \pmod{4}$, then there is a $\gamma_t$-set of $P_p$ that contains one of its end vertices.

**Theorem 1.47** [4].

If $G$ is a connected graph with $p \geq 3$ vertices, then $\gamma_t(G) \leq \frac{2p}{3}$.

**Theorem 1.48** [4].

If a graph $G$ has no isolated vertices, $\gamma_t(G) \leq p - \Delta(G) + 1$. 

20
Theorem 1.49[4].
If a graph $G$ is connected, then $\gamma_r(G) \leq p - \Delta(G)$.

Theorem 1.50[4].
If a graph $G$ and $G^c$ have no isolated vertices, $\gamma_r(G) + \gamma_r(G^c) \leq p + 2$ with equality if and only if $G$ or $G^c$ is $mK_2$.

Theorem 1.51[2].
A clique partition of $G$ is a partition of $V(G)$ into non-empty subsets $V_1, V_2, \ldots, V_k$ such that each graph induced by $V_i$ $(1 \leq i \leq k)$ is a complete subgraph of $G$. The clique partition number, \( \chi_c(G) \), of $G$ is the smallest number of partitions for which there is a clique partition of $G$.

Definition 1.52[1].
The uniform domination number $\gamma_u(G)$, is the least positive integer $k$ such that any $k$-element subset of $V$ is a dominating set of $G$.

Theorem 1.53[1].
For any graph $G$, $\gamma_u(G) = p - \delta(G)$.

The concept of safe clique partition number and the dominator chromatic number of $G$ were introduced by Gera et. al [8].

Definition 1.54[8].
A dominator coloring of $G$ is a proper coloring of $G$ with the additional property that every vertex in $G$ dominates an entire color class. The dominator chromatic number, $\chi_d(G)$ is the minimum number of colors for which there exists a dominator coloring of $G$.

**Definition 1.55[8].**

A clique partition of $G$ is a partition of $V(G)$ into non empty subsets $V_1, V_2, ..., V_r$ such that each sub graph induced by $V_i (1 \leq i \leq r)$ is a complete sub graph of $G$. The clique partition number, $\chi(G)$ is the smallest number of partitions for which there exists a clique partition of $G$.

**Definition 1.56[8].**

A safe clique partition of a graph $G$ is a partition of the vertices of $G$ into complete sub graphs with the additional property that for each vertex $v$, there exists a complete sub graph that has no vertex in the open neighborhood of $v$. The smallest number of partitions for which there exists a safe clique partition of $G$ is called the safe clique partition number of $G$ and is denoted by $\chi_s(G)$.

**Theorem 1.57[8].**

$$1 + \left\lceil \frac{p}{3} \right\rceil \quad ; \quad p = 2, 3, 4, 5, 7$$

The path $P_p$ of order $p \geq 2$ has $\chi_d(P_p) = \left\lceil \frac{p}{3} \right\rceil$; otherwise.
Lemma 1.58[8].

Let G be a tree of order $p \geq 5$. A maximum matching $M^*$ of G is unsafe if and only if G is the wounded spider $W_{a,b}(a \geq 1, b \geq 0)$.

Theorem 1.59[9].

Let G be any connected graph. Then $\max \{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$.

Theorem 1.60[8].

If G is a caterpillar in which the vertices of degree less than 3 are independent, and if the spine of G contains exactly $r$ vertices of degree at least 3, then $\chi_d(G) = r + 1$.

Definition 1.61[11]

Let H be a non trivial proper sub graph of G. Two vertices u and v (two lines x and y) are said to be H-adjacent if there exists a sub graph $H'$ of G which is isomorphic to H such that $H'$ contains u and v (x and y).

Definition 1.62[11]

The open H- neighborhood set of a vertex v of G is the set of all vertices of G which are H-adjacent to v and is denoted by $N_H(v)$ and the closed H-neighborhood set of v is $N_H[v] = N_H(v) \cup \{v\}$.

Definition 1.63[11]

The H-degree of a vertex v denoted by $\deg_H(v)$, and is defined by $\deg_H(v) = |N_H(v)|$. The maximum H-degree of G is denoted by $\Delta_H(G)$, and is defined by
max \{ |N_H(v)| : v \in V(G) \} and the minimum H-degree of G is denoted by \( \delta_H(G) = \{ |N_H(v)| : v \in V(G) \} \). If a vertex v is not H-adjacent to any vertex of G, then v is H-isolated in G.