Total Dominator Colorings in Paths, Cycles and Caterpillars
Chapter-5

Total Dominator Colorings in Paths, Cycles and Caterpillars

Abstract

Let $G$ be a graph without isolated vertices. A total dominator coloring of a graph $G$ is a proper coloring with the additional property that every vertex in $G$ properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of $G$ is called the total dominator chromatic number of $G$ and is denoted by $\chi_{td}(G)$. In this chapter we determine the total dominator chromatic number in paths, cycles and caterpillars.

This concept is inspired by total domination just as dominator coloring is inspired by domination. Unless otherwise specified, $n$ denotes an integer greater than or equal to 2.
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Notation 5.1. Usually, the vertices of $P_n$ are denoted by $u_1, u_2, ..., u_n$ in order. We also denote a vertex $u_i \in V(P_n)$ with $i > \left\lceil \frac{n}{2} \right\rceil$ by $u_{(n-i)}$. For example, $u_{n-1}$ by $u_{-2}$.

This helps us to visualize the position of the vertex more clearly.

Notation 5.2. For $i < j$, we use the notation $(i, j)$ for the sub path induced by $\langle u_i, u_{i+1}, ..., u_j \rangle$. For a given coloring $C$ of $P_n$, $C/([i, j])$ refers to the coloring $C$ restricted to $([i, j])$.

Definition 5.3. We know from theorem (4.8) that $\chi_{td}(P_n) = \{g(P_n), g(P_n) + 1, g(P_n) + 2\}$. We call the integer $n$, good (respectively bad, very bad) if $\chi_{td}(P_n) = g(P_n) + 1$, respectively $\chi_{td}(P_n) = g(P_n)$.

Section 5.1

Determination of $\chi_{td}(P_n): n \geq 2$.

First we note the values of $\chi_{td}(P_n)$ for small $n$. Some of these values are computed in Theorems 5.10 and 5.11 and the remaining can be computed similarly.
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Thus $n = 2, 3, 6$ are very bad integers and we shall show that these are the only bad integers. First, we prove a result which shows that for large values of $n$, the behavior of $\chi_{td}(P_n)$ depends only on the residue class of $n \mod 4$ [More precisely, if $n$ is good, $m > n$ and $m \equiv n \pmod{4}$ then $m$ is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their respective residue classes. This therefore classifies the good integers.

**Fact 5.4.** Let $1 < i < n$ and let $C$ be a td-coloring of $P_n$. Then, if either $u_i$ has a repeated color or $u_{i+2}$ has a non-repeated color, $C/\langle[i+1, n] \rangle$ is also a td-coloring.

This fact is used extensively in this chapter.

**Lemma 5.5.** $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2$. 

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<th>$n$</th>
<th>$\gamma_1(P_n)$</th>
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Proof. For \(2 \leq n \leq 5\), this is directly verified from the table.

We may assume \(n \geq 6\).

Let \(u_1, u_2, u_3, \ldots, u_{n+4}\) be the vertices of \(P_{n+4}\) in order.

Let \(C\) be a minimal td-coloring of \(P_{n+4}\).

Clearly, \(u_2\) and \(u_3\) are non-repeated colors.

First suppose \(u_4\) is a repeated color.

Then \(C / \langle[5, n+4]\rangle\) is a td-coloring of \(P_n\). Further, \(C / \langle[1, 4]\rangle\) contains at least two color classes of \(C\).

Thus \(\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2\).

Similarly the result follows if \(u_3\) is a repeated color.

Thus we may assume \(u_4\) and \(u_3\) are non-repeated colors.

But the \(C / \langle[3, n+2]\rangle\) is a td-coloring and since \(u_2\) and \(u_3\) are non-repeated colors, we have in this case also \(\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2\).

Corollary 5.6. If for any \(n\), \(\chi_{td}(P_n) = \gamma_1(P_n) + 2\), \(\chi_{td}(P_m) = \gamma_1(P_m) + 2\) for all \(m > n\) with \(m \equiv n(\text{mod} 4)\).

Proof. By lemma 5.5, \(\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2\)

\[
= \gamma_1(P_n) + 2 + 2
\]

\[
= \gamma_1(P_{n+4}) + 2.
\]
Corollary 5.7. No integer \( n \geq 7 \) is a very bad integer.

Proof. For \( n = 7, 8, 9, 10 \), this is verified from the table. The result then follows from the Lemma 5.5.

Corollary 5.8. The integers 2, 3, 6 are the only very bad integers.

Next, we show that \( n = 8, 13, 15, 22 \) are good integers. In fact, we determine \( \chi_{td}(P_n) \) for small integers and also all possible minimum td-colorings for such paths. These ideas are used more strongly in determination of \( \chi_{td}(P_n) \) for \( n = 8, 13, 15, 22 \).

Definition 5.9. Two td-colorings \( C_1 \) and \( C_2 \) of a given graph \( G \) are said to be equivalent if there exists an automorphism \( f : G \rightarrow G \) such that \( C_2(v) = C_1(f(v)) \) for all vertices \( v \) of \( G \). This is clearly an equivalence relation on the set of td-colorings of \( G \).

Theorem 5.10. Let \( V(P_n) = \{u_1, u_2, \ldots, u_n\} \) as usual. Then

(i) \( \chi_{td}(P_2) = 2 \). The only minimum td-coloring is (given by the color classes)

\[ \{\{u_1\}, \{u_2\}\} \]

(ii) \( \chi_{td}(P_3) = 2 \). The only minimum td-coloring is \( \{\{u_1, u_3\}, \{u_2\}\} \).
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(iii) $\chi_{td}(P_4) = 3$ with unique minimum coloring $\{\{u_1, u_3\}, \{u_2\}, \{u_5\}\}$.

(iv) $\chi_{td}(P_5) = 4$. Any minimum coloring is equivalent to one of

$\{\{u_1, u_2\}, \{u_3\}, \{u_4\}\}$ or $\{\{u_1, u_3\}, \{u_2\}, \{u_4\}\}$ or

$\{\{u_2, u_3\}, \{u_4\}, \{u_5\}\}$

(v) $\chi_{td}(P_6) = 4$ with unique minimum coloring $\{\{u_1, u_3\}, \{u_4, u_6\}, \{u_2\}, \{u_5\}\}$.

(vi) $\chi_{td}(P_7) = 5$. Any minimum coloring is equivalent to one of

$\{\{u_1, u_3\}, \{u_4\}, \{u_5\}, \{u_6\}\}$ or $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}, \{u_6\}\}$ or

$\{\{u_1, u_4, u_7\}, \{u_2\}, \{u_3\}, \{u_6\}\}$

Proof. We prove only (vi). The rest are easy to prove.

Now, $\chi_t(P_7) = \left\lceil \frac{7}{2} \right\rceil = 4$.

Clearly $\chi_{td}(P_7) \geq 4$.

We first show that $\chi_{td}(P_7) \neq 4$.

Let $C$ be a td-coloring of $P_7$ with 4 colors.

The vertices $u_2$ and $u_3 = u_6$ must have non-repeated colors.

Suppose now that $u_3$ has a repeated color.

Then $\{u_1, u_2, u_3\}$ must contain two color classes and $C/\langle[4, 7]\rangle$ must be a td-coloring which will require at least 3 new colors (by (iii)).

Hence $u_3$ and similarly $u_3$ must be non-repeated colors.

But, then we require more than 4 colors. Thus $\chi_{td}(P_7) = 5$. 

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Let $C$ be a minimal td-coloring of $P_7$.

Let $u_2$ and $u_\alpha$ have colors 1 and 2 respectively.

Suppose that both $u_3$ and $u_\alpha$ are non-repeated colors. Then, we have the coloring $\{\{u_1,u_4,u_7\},\{u_2\},\{u_3\},\{u_5\},\{u_6\}\}$. If either $u_3$ or $u_\alpha$ is a repeated color, then the coloring $C$ can be verified to be equivalent to the coloring given by $\{\{u_1,u_4\},\{u_2\},\{u_3\},\{u_5\},\{u_6\}\}$ or $\{\{u_1,u_4\},\{u_2\},\{u_3\},\{u_5,u_7\},\{u_6\}\}$. [1]

We next show that $n = 8, 13, 15, 22$ are good integers.

**Theorem 5.11.** $\chi_{td}(P_n) = \gamma_r(P_n) + 2$ if $n = 8, 13, 15, 22$.

**Proof.** As usual, we always adopt the convention $V(P_n) = \{u_1, u_2, \ldots, u_n\}$; $u_\alpha = u_{n+\alpha}$ for $i \geq \left\lceil \frac{n}{2} \right\rceil$; $C$ denotes a minimum td-coloring of $P_n$.

We have only to prove $|C| > \gamma_r(P_n) + 1$.

We have the following four cases.

**Case(i).** $n = 8$.

Let, if possible $|C| = 5$.

Then, as before $u_2$, being the only vertex dominated by $u_1$ has a non-repeated color.

The same argument is true for $u_\alpha$ also.
If now $u_3$ has a repeated color, $\{u_1, u_2, u_3\}$ contains 2-color classes. As $C/\langle[4, 8]\rangle$ is a td-coloring, we require at least 4 more colors.

Hence, $u_3$ and similarly $u_3$ must have non-repeated colors.

Thus, there are 4 singleton color classes $\{u_2\}, \{u_3\}, \{u_2\}$ and $\{u_3\}$.

The two adjacent vertices $u_4$ and $u_4$ contribute two more colors.

Thus $|C|$ has to be 6.

Case(ii). $n = 13$

Let, if possible $|C| = 8 = \gamma_t(P_{13}) + 1$.

As before $u_2$ and $u_2$ are non-repeated colors. Since $\chi_{td}(P_{10}) = 7 + 2 = 9$, $u_3$ can not be a repeated color, arguing as in case (i). Thus, $u_3$ and $u_3$ are also non-repeated colors.

Now, if $u_1$ and $u_1$ have different colors, a diagonal of the color classes chosen as $\{u_1, u_1, u_2, u_2, u_2, u_3, u_3, \ldots\}$ form a totally dominating set of cardinality $8 = \gamma_t(P_{13}) + 1$.

However, clearly $u_1$ and $u_1$ can be omitted from this set without affecting total dominating set giving $\gamma_t(P_{13}) \leq 6$, a contradiction. Thus, $u_1$ and $u_1 = u_1$ have the same color say 1.

Thus, $\langle[4, -4]\rangle = \langle[4, 10]\rangle$ is colored with 4 colors including the repeated color 1.

Now, each of the pair of vertices $\{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_10\}$ contains a color classes.
Thus $u_5 = u_5$ must be colored with 1. Similarly, $u_3$. Now, if $\{u_4, u_6\}$ is not a color class, the vertex with repeated color must be colored with 1 which is not possible, since an adjacent vertex $u_3$ which also has color 1.

Thus $\{u_4, u_6\}$ is a color class.

Similarly $\{u_8, u_{10}\}$ is also a color class.

But then, $u_7$ will not dominate any color class.

Thus $|C| = 9$.

**Case (iii).** $n = 15$

Let, if possible $|C| = 9$.

Arguing as before, $u_2, u_2', u_3$ and $u_3$ have non-repeated colors $[\chi_{\mathbb{N}}(P_{12}) = 8]$; $u_1$ and $u_4$ have the same color, say 1. The section $[4, -4] = [4, 12]$ consisting of 9 vertices is colored by 5 colors including the color 1. An argument similar to the one used in case (ii), gives $u_4$ (and $u_4$) must have color 1. Thus, $C/([5, -5])$ is a td-coloring with 4 colors including 1.

Now, the possible minimum td-coloring of $P_7$ are given by theorem 5.10.

We can check that 1 can not occur in any color class in any of the minimum colorings given. e.g. take the coloring given by $\{u_5, u_6\}, \{u_6\}, \{u_7\}, \{u_9, u_{11}\}, \{u_{10}\}$.

If $u_6$ has color 1, $u_5$ can not dominate a color class. Since $u_4$ has color 1, $\{u_5, u_6\}$ can not be color class 1 and so on.
Thus $\chi_{sd}(P_{15}) = 10$.

Case (iv). $n = 22$.

Let, if possible, $|C| = \gamma_r(P_{22}) + 1 = 13$.

We note that $\chi_{sd}(P_{19}) = \gamma_r(P_{19}) + 2 = 12$.

Then, arguing as in previous cases, we get the following facts.

Fact 1. $u_2, u_{-2}, u_3, u_{-3}$ have non-repeated colors.

Fact 2. $u_1$ and $u_{-1}$ have the same color, say 1.

Fact 3. $u_7$ is a non-repeated color.

This follows from the facts, otherwise $C/\langle [8,22] \rangle$ will be a td-coloring; The section $\langle [1,7] \rangle$ contain 4 color classes which together imply $\chi_{sd}(P_{22}) \geq 4 + \chi_{sd}(P_{15}) = 4 + 10 = 14$.

In particular $\{u_5, u_7\}$ is not a color class.

Fact 4. The facts 1 and 2, it follows that $C/\langle [4,-4] \rangle = C/\langle [4,19] \rangle$ is colored with 9 colors including 1.

Since each of the pair $\{ \{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_{10}\}, \{u_9, u_{11}\}, \{u_{12}, u_{14}\}, \{u_{13}, u_{15}\}, \{u_{16}, u_{18}\}, \{u_{17}, u_{19}\} \}$ contain a color class, if any of these pairs is not a color class, one of the vertices must have a non-repeated color and the other colored with 1.
From fact 3, it then follows that the vertex \( u_5 \) must be colored with 1. It follows that \( \{u_4, u_6\} \) must be a color class, since otherwise either \( u_4 \) or \( u_6 \) must be colored with 1.

Since \( \{u_4, u_6\} \) is a color class, \( u_7 \) must dominate the color class \( \{u_6\} \).

We summarize:
* \( u_2, u_3, u_7, u_8 \) have non-repeated colors.
* \( \{u_4, u_6\} \) is a color class
* \( u_4 \) and \( u_6 \) are colored with color 1.

Similarly,
* \( u_2, u_3, u_7, u_8 \) have non-repeated colors.
* \( \{u_4, u_6\} \) is a color class.
* \( u_4 \) and \( u_6 \) are colored with color 1.

Thus the section \( \langle [9,-9] \rangle = \langle [9,14] \rangle \) must be colored with 3 colors including 1. This is easily seen to be not possible, since for instance this will imply both \( u_{13} \) and \( u_{14} \) must be colored with color 1. Thus, we arrive at a contradiction.

Thus \( \chi_{id}(P_{22}) = 14. \]
(iii) Any integer of the form $4k+2$, $k \geq 5$ is good.

(iv) Any integer of the form $4k+3$, $k \geq 3$ is good.

The integers $n = 2, 3, 6$ are very bad and $n = 4, 5, 7, 9, 10, 11, 14, 18$ are bad.

Remark 5.13. Let $C$ be a minimal td-coloring of $G$. We call a color class in $C$, a non-dominated color class (n-d color class) if it is not dominated by any vertex of $G$. These color classes are useful because we can add vertices to these color classes without affecting td-coloring.

Lemma 5.14. Suppose $n$ is a good number and $P_n$ has a minimal td-coloring in which there are two non-dominated color class. Then the same is true for $n+4$ also.

Proof. Let $C_1, C_2, \ldots, C_r$ be the color classes for $P_n$ where $C_1$ and $C_2$ are non-dominated color classes.

Suppose $u_n$ does not have color $C_1$.

Then $C_1 \cup \{u_{n+1}\}, C_2 \cup \{u_{n+4}\}, \{u_{n+2}\}, \{u_{n+3}\}, C_3, C_4, \ldots, C_r$ are required color classes for $P_{n+4}$.

ie. we add a section of 4 vertices with middle vertices having non-repeated colors and end vertices having $C_1$ and $C_2$ with the coloring being proper.

Further, suppose the minimum coloring for $P_n$, the end vertices have different colors.
Then the same is true for the coloring of $P_{n+4}$ also.

If the vertex $u_i$ of $P_n$ does not have the color $C_2$, the new coloring for $P_{n+4}$ has this property.

If $u_i$ has color $C_2$, then $u_n$ does not have the color $C_2$.

∴ We can take the first two color classes of $P_{n+4}$ as $C_1 \cup \{u_{n+4}\}$ and $C_2 \cup \{u_{n+4}\}$.

**Corollary 5.15.** Let $n$ be a good number. Then $P_n$ has a minimal td-coloring in which the end vertices have different colors. [It can be verified that the conclusion of the corollary is true for all $n \neq 3, 4, 11$ and 18].

**Proof.** We claim that $P_n$ has a minimum td-coloring in which

(i) There are two non-dominated color classes.

(ii) The end vertices have different colors.

For $n = 8, 13, 15, 22$. 

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Figure 5.1

Now, it follows from the lemma (5.14) that (i) and (ii) are true for every good integer. ■

Corollary 5.16. Let \( n \) be a good integer. Then, \( \exists \) a minimum td-coloring for \( P_n \) with two \( n \)-d color classes.

Section 5.2

In this section we determine \( \chi_{td}(C_n) \).

It is trivially true that \( \chi_{td}(C_3) = 3 \) & \( \chi_{td}(C_4) = 2 \).

We assume \( n \geq 5 \).

Lemma 5.17. If \( P_n \) has a minimum td-coloring in which the end vertices have different colors, then \( \chi_{td}(C_n) \leq \chi_{td}(P_n) \).

Proof. Join \( u_t u_n \) by an edge & we get an induced td-coloring of \( C_n \). ■

Corollary 5.18. \( \chi_{td}(C_n) \leq \chi_{td}(P_n) \) \( \forall n \neq 3, 11, 18 \).
Lemma 5.19. If $C_n$ has a minimal td-coloring in which either $\exists$ a color class of the form $N(x)$, where $x$ is a non-repeated color or no color class of the form $N(x)$, then $\chi_{td}(P_n) \leq \chi_{td}(C_n)$.

Proof. We have assumed $n > 3$. If $n = 3$, conclusion is trivially true.

We have the following two cases.

Case (1). $C_n$ has a minimal td-coloring $C$ in which there is a color class of the form $N(x)$, where $x$ is a non-repeated color.

Let $C_n$ be the cycle $u_1u_2 \ldots u_n u_1$.

Let us assume $x = u_2$ has a non-repeated color, and $N(x) = \{u_1, u_2\}$ is the color class of color $r_1$.

Then $u_{n-1}$ has a non-repeated color since $u_n$ has to dominate a color class which must be contained in $N(u_n) = \{u_1, u_{n-1}\}$.

Thus $C/\{1, n\}$ is a td-coloring.

Thus $\chi_{td}(P_n) \leq \chi_{td}(C_n)$.

Case (2). $\exists C_n$ has a minimal td-coloring which has no color class of the form $N(x)$.

It is clear from the assumption that any vertex with a non-repeated color has an adjacent vertex with non-repeated color.

We consider two sub cases.

Sub case (a). There are two adjacent vertices $u, v$ with repeated color.
Then the two vertices on either side of \( u, v \) say \( u_i \) and \( v_i \) must have non-repeated colors.

Then the removal of the edge \( uv \) leaves a path \( P_n \) and \( C/[\{1, n\}] \) is a td-coloring.

**Sub case (b).** There are adjacent vertices \( u, v \) with \( u \) (respectively \( v \)) having repeated (respectively non-repeated) color.

Then consider the vertex \( u_i(\neq v) \) adjacent to \( u \).

We may assume \( u_i \) has non-repeated color (because of sub case (a)). \( v_i \) must also have a non-repeated color since \( v \) must dominate a color class \& \( u \) has a repeated color. Once again, \( C/(C_n-uv) \) is a td-coloring \& the proof is as in sub case (a).

Since either sub case (a) or sub case (b) must hold, the lemma follows.

**Lemma 5.20.** \( \chi_{td}(C_n) = \chi_{td}(P_n) \) for \( n = 8, 13, 15, 22. \)

**Proof.** We prove for \( n = 22. \)

By Lemma 5.17, \( \chi_{td}(P_{22}) \geq \chi_{td}(C_{22}). \)

Let if possible \( \chi_{td}(C_{22}) < \chi_{td}(P_{22}) = 14. \)

Then by Lemma 5.19, \( C_{22} \) has a minimal td-coloring in which there is a color class of the form \( N(x) \), where \( x \) is a repeated color (say \( C_i \)).

Suppose \( x = u_2 \)

First, we assume that the color class of \( u_2 \) is not \( N(u_i) \) or \( N(u_s) \).

Then we have \( u_4, u_5, u_{22}, u_{21} \) must be non-repeated colors.
Then $C/\{[6,20]\}$ is a coloring (which may not be a td-coloring for the section) with 8 colors including $C_1$.

$\Rightarrow$ The vertices $u_7$ and $u_{19}$ have the color $C_1$.

(The sets $\{u_6, u_8\}, \{u_7, u_9\}, \{u_{10}, u_{12}\}, \{u_{11}, u_{13}\}, \{u_{14}, u_{16}\}, \{u_{15}, u_{17}\}, \{u_{18}, u_{20}\}$ must contain color classes. $\therefore$ The remaining vertex $u_{19}$ must have color $C_1$. Similarly, going the other way, we get $u_7$ must have color $C_1$.)

Then $\{u_6, u_8\}, \{u_{18}, u_{20}\}$ are color classes and $u_9, u_{10}, u_{16}, u_{17}$ are non-repeated colors.

This leads $\{[11,15]\}$ to be colored with 2 colors including $C_1$, which is not possible.
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Hence $\chi_{td}(C_{22}) = 14 = \chi_{td}(P_{22})$.

If the color class of $u_2$ is $N(u_2)$ or $N(u_3)$, the argument is similar.

Proof is similar for $n = 8, 13, 15$.\]

**Lemma 5.21** Let $n$ be a good integer. Then $\chi_{td}(P_n) \leq \chi_{td}(C_n)$

**Proof.** We prove the lemma by induction on $n$

Let $u_1, u_2, ..., u_n$ be vertices of $C_n$ in order.

Let $C$ be a minimal td-coloring of $C_n$.

For the least good integers in their respective residue classes mod 4 is 8, 13, 15, 22, the result is proved in the previous lemma 5.20. So we may assume that the result holds for all good integers $< n$ & that $n - 4$ is also a good integer.

First suppose, $\exists$ a color class of the form $N(x)$

Let $x = u_2$.

Suppose $u_2$ has a repeated color

Then we have $u_4, u_5, u_n, u_{n-1}$ must be non-repeated color.

We remove the vertices $\{u_1, u_2, u_3, u_n\}$ and add an edge $u_4u_{n-1}$ in $C_n$.

\[ \therefore \text{we have the coloring } C/\langle 4, n-1 \rangle \text{ is a td-coloring with colors } \chi_{td}(C_n) - 2. \]

\[ \therefore \chi_{td}(C_n) \geq 2 + \chi_{td}(C_{n-4}) \]

\[ \geq 2 + \chi_{td}(P_{n-4}) \]

\[ = \chi_{td}(P_n). \]
Figure 5.3

If \( x \) is a non-repeated color, then by lemma 5.19, \( \chi_{ad}(P_n) \leq \chi_{ad}(C_n) \).

If there is no color class of the form \( N(x) \), then \( \chi_{ad}(P_n) \leq \chi_{ad}(C_n) \). □

Theorem 5.22. \( \chi_{ad}(C_n) = \chi_{ad}(P_n) \), for all good integers \( n \).

Proof. The result follows from corollary 5.18 and Lemmas 5.20, 5.21.

Remark 5.23. Thus the \( \chi_{ad}(C_n) = \chi_{ad}(P_n) \) for \( n = 8, 12, 13, 15, 16, 17 \) and \( \forall n \geq 19 \).

It can be verified that \( \chi_{ad}(C_n) = \chi_{ad}(P_n) \) for \( n = 5, 6, 7, 9, 10, 14 \) and that \( \chi_{ad}(C_n) = \chi_{ad}(P_n)+1 \) for \( n = 3, 11, 18 \) and that \( \chi_{ad}(P_n) = \chi_{ad}(C_4)+1 \).
Section 5.3. Total dominator colorings in caterpillars.

After the classes of stars and paths, caterpillars are perhaps the simplest class of trees. For this reason, for any newly introduced parameter, we try to obtain the value for this class. e.g. $\chi_d$ has been calculated for a restricted class of caterpillars in [8]. In this section, we give an upper bound for $\chi_d(T)$, where $T$ is a caterpillar (with some restriction). First, we prove a theorem for a very simple type which however illustrates the ideas to be used in the general case [Theorem 2.25].

Theorem 5.24. Let $G$ be a caterpillar such that

(i) No two vertices of degree two are adjacent

(ii) The end vertebrae have degree at least 3.

(iii) No vertex of degree 2 is a support vertex.

Then $\chi_d(G) \leq \left\lceil \frac{3r + 2}{2} \right\rceil$

Proof.

Let $C$ be the spine of $G$. Let $u_1, u_2, \ldots, u_r$ be the support vertices and $u_{r+1}, u_{r+2}, \ldots, u_{2r-1}$ be the vertices of degree 2 in $C$.

In a td-coloring of $G$, all support vertices receive a non-repeated color, say 1 to $r$ and all pendant vertices receive the same repeated color say $r+1$ & the vertices $u_{r+1}$ and $u_{2r-1}$ receive a non-repeated color say $r+2$ and $r+3$ respectively.

Consider the vertices $\{u_{r+2}, u_{r+3}, \ldots, u_{2r-2}\}$. 

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We consider the following two cases.

Case 1. \( r \) is even.

In this case the vertices \( u_{r+3}, u_{r+5}, \ldots, u_{r\left(\frac{r+2}{2}\right)} \), \( u_{r+8}, u_{r+10}, \ldots, u_{r\left(\frac{r+2}{2}\right)} \), \( u_{2r-3} \) receive the non-repeated colors say \( r + 4 \) to \( r + \left(\frac{r+2}{2}\right) = \frac{3r+2}{2} \) and the remaining vertices \( u_{r+2}, u_{r+4}, \ldots, u_{2r-2} \) receive the already used repeated color \( r+1 \) respectively.

Thus \( \chi'_d(G) \leq \frac{3r+2}{2} \).

Case 2. \( r \) is odd.

In this case the vertices \( u_{r+3}, u_{r+5}, \ldots, u_{r\left(\frac{r+2}{2}\right)} \), \( u_{r+8}, u_{r+10}, \ldots, u_{r\left(\frac{r+2}{2}\right)} \), \( u_{2r-4}, u_{2r-2} \) receive the non-repeated colors say \( r + 4 \) to \( r + \left(\frac{r+3}{2}\right) = \frac{3r+3}{2} \) and the remaining vertices \( u_{r+2}, u_{r+4}, \ldots, u_{2r-3} \) receive the already used repeated color \( r+1 \) respectively.

Thus \( \chi'_d(G) \leq \frac{3r+3}{2} = \left\lceil \frac{3r+2}{2} \right\rceil \).

Illustration 5.25.
Let $G$ be a caterpillar of class 2 having exactly $r$ vertices of degree at least 3 and $r_i$ zero strings of length $i$, $2 \leq i \leq m$, $m =$ maximum length of a zero string in $G$. Further suppose that $r_n \neq 0$ for some $n$, where $n-2$ is a good number and that end vertebrae are of degree at least 3. Then $\chi_{td}(G) \leq 2(r+1) + \sum_{i=3}^{m} r_i \left\lceil \frac{i-2}{2} \right\rceil + \sum_{i=4}^{m} r_i \left( \left\lfloor \frac{i-2}{2} \right\rfloor + 1 \right)$.

Proof.

\[ \chi_{td}(G) = 12 = \left\lceil \frac{3r+2}{2} \right\rceil \]
Let $S$ be the spine of the caterpillar $G$ and let $V(S) = \{u_1, u_2, \ldots, u_r\}$.

We give the coloring of $G$ as follows:

Vertices in $S$ receive non-repeated colors, say from 1 to $r$. The set $N(u_i)$ is given the color $r + j, 1 \leq j \leq r$ ($u_i$ is not adjacent to an end vertex of zero string of length 3 and if a vertex is adjacent to two supports, it is given one of the two possible colors).

This coloring takes care of any zero string of length 1 or 2.

Now, we have assumed $r_n \neq 0$ for some $n$, where $n - 2$ is a good number. Hence there is a zero string of length $n$ in $G$.

By corollary 5.16, there is a minimum $td$-coloring of this path in which there are two $n$-d colors. We give the sub path of length $n$ this coloring with $n$-d colors being denoted by $2r + 1, 2r + 2$.

The idea is to use these two colors whenever $n$-d colors occur in the coloring of zero strings. Next, consider a zero string of length 3, say

![Figure 5.5](image.png)

where $u_i$ and $u_{i+1}$ are vertices of degree at least 3 and we have denoted the vertices of the string of length 3 by $x_1, x_2, x_3$ for simplicity.
Then, we give $x_1$ or $x_3$, say $x_1$ with a non-repeated color; we give $x_2$ and $x_3$ the colors $2r + 1$ and $2r + 2$ respectively.

Thus each zero string of length 3 introduces a new color and $\left\lceil \frac{3-2}{2} \right\rceil = 1$.

Similarly, each zero string of length $i$ introduces $\left\lceil \frac{i-2}{2} \right\rceil$ new colors when $i = 1, 2, 3 \text{ (mod 4)}$. However, the proof in cases when $i > 3$ is different from case $i = 3$ (but are similar in all such cases in that we find a td-coloring involving two n-d colors).

E.g. a zero string of length 11.

We use the same notation as in case $i = 3$ with a slight difference:

```
\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.6.png}
\caption{Figure 5.6}
\end{figure}
```

$u_i$ and $u_{i+1}$ being support vertices receive colors $i$ and $i + 1$. $x_i$ and $x_{i+1}$ receive $r + i$ and $r + i + 1$ respectively.

For the coloring of $P_5$, we use the color classes $\{y_1, y_4\}$, $\{y_2\}$, $\{y_3\}$, $\{y_5, y_9\}$, $\{y_6\}$, $\{y_7\}$, $\{y_8\}$.

We note that this is not a minimal td-coloring which usually has no n-d color classes.
This coloring has the advantage of having two n-d color classes which can be
given the class $2r + 1$ and $2r + 2$ and the remaining vertices being given non-
repeated colors.

In cases where $i$ is a good integer, $P_{r-2}$ requires $\left\lfloor \frac{i-2}{2} \right\rfloor + 2$ colors. However there
will be two n-d color classes for which $2r + 1$ and $2r + 2$ can be used.

Thus each such zero string will require only $\left\lfloor \frac{i-2}{2} \right\rfloor$ new colors (except for the
path containing the vertices we originally colored with $2r + 1$ and $2r + 2$).

However, if $i \equiv 0 \pmod{4}$, $i-2 \equiv 2 \pmod{4}$ and we will require $\left\lfloor \frac{i-2}{2} \right\rfloor + 1$ new
colors.

It is easily seen this coloring is a td-coloring.

Hence the result.\[\]  

Illustration 5.26.
Chapter 5: Total Dominator Colorings in Paths, Cycles and Caterpillars

\[ \chi_{td}(T_1) = 12 < 2(r+1) + r_1 \left\lceil \frac{10-2}{2} \right\rceil \]

(1) The condition that end vertices are of degree at least 3 is adopted for the sake of simplicity. Otherwise the caterpillar 'begins' or 'ends' (or both) with a vertex adjacent to only one other vertex. In this case, the \( \chi_{td} \) value for this graph will be

\[ \chi_{td}(G) = r + 1 \]

(2) It is apparent from the second graph that the bound of Theorem 5.26 does not appear to be tight. We find that the correct bound will have \( 2r + 3 \) on the right in place of \( 2r + 2 \). There are graphs

\[ \chi_{td}(T_2) = 15 = 2(r+1) + r_3 \left\lceil \frac{3-2}{2} \right\rceil + r_{10} \left\lceil \frac{10-2}{2} \right\rceil \]

(3) The bound in theorem 5.26 does not appear to be tight. We find that the correct bound will have \( 2r + 3 \) on the right instead of \( 2r + 2 \). There are graphs

\[ \chi_{td}(T_3) = 17 = 2(r+1) + r_3 + r_{12} \left\lceil \frac{12-2}{2} \right\rceil + 1 \]

Figure 5.7
Remark 5.27.

(1) The condition that end vertebrae are of degree at least 3 is adopted for the sake of simplicity. Otherwise the caterpillar ‘begins’ or ‘ends (or both) with a segment of a path and we have to add the $\chi_d$-values for this (these) path(s).

(2) If in theorem 5.24, we assume that all the vertices of degree at least 3 are adjacent (instead of (ii)), we get $\chi_d(G) = r + 1$.

(3) The bound in theorem 5.26 does not appear to be tight. We feel that the correct bound will have $2r + 1$ on the right instead of $2r + 2$. There are graphs which attain this bound.