Chapter 5

Quasiperiodic behavior of ion-acoustic waves in electron-positron-ion magnetoplasmas

5.1 Introduction

During last few decades, the nonlinear dynamics of ion-acoustic waves has been studied both theoretically and experimentally [1]-[2]. Ref. [3] studied ion-acoustic waves experimentally for the first time. It is to be noted that electron-positron (e-p) pairs exist in plasmas emanating both from the pulsars and from the inner region of the accretion discs surrounding the central black hole in active galactic nuclei [4]. Electron-positron pairs are also observed in the Van Allen radiation belt and near the polar cap of fast rotation neutron stars [5], in semiconductor plasma[6], in intense laser fields [7], in tokamaks [8], and in the solar atmosphere [9]. It is important to note
that the e-p plasmas behave significantly different from the typical electron-ion (e-i) plasmas [10]-[11]. It has been observed that there exists always a small number of ions with electrons and positrons in astrophysical environments. That is why it is important to study the linear and nonlinear dynamics of plasma waves in electron-positron-ion (e-p-i) plasmas. A number of important research investigations have been performed to study the linear and nonlinear wave features in e-p and e-p-i plasmas [12]-[14]. Recently, using Maxwellian assumption, many researchers studied the nonlinear propagation of ion acoustic waves in e-p-i plasmas [15]-[16]. The study of nonlinear propagation of radiation in e-p plasmas is important because of high effective temperatures of pulsar radio emission [17] and a feasible source of the large frequency shifts in such emissions [18]. Based on the space plasma observation, and theoretical investigations [19]-[20], one can easily guess the presence of ion and electron populations that are far away from their thermodynamic equilibrium.

There are some nonlinear evolution equations such as the Sine-Gordon, the KdV and the nonlinear Schrodinger equations which arise from many physical fields and they are Hamiltonian integrable [21]-[22]. But, it is very much important to note that the integrability could be destroyed by the effect of external perturbations which occur in some real physical environments[23]-[25]. The nature of this external perturbation may be different and varies from one physical situation to others. So, a significant attention is recently made to the study of nonlinear wave equations in the presence of external perturbations. It is also well known that a completely integrable nonlinear wave equations can not describe chaotic behavior. But, the addition of an external perturbation to a nonlinear integrable wave equation may lead to chaotic dynamics. It is also important to note that the nonlinear evolution equations (KdV, KP, ZK) can not describe the quasi-periodic behavior in plasmas. Recently, ref. [26] observed the
quasi-periodic behavior in quantum plasmas due to the presence of bohm potential term $\frac{H^2}{2\sqrt{n}} \frac{\partial^2}{\partial x^2} (\sqrt{n})$, where $H$ is quantum parameter and $n$ is the number density of hot electron. But, there is no attempt to study the quasi-periodic behavior in classical plasmas to the best of our knowledge. So, it is important to study the quasi-periodic behavior of the nonlinear wave equations (KdV, KP, ZK) in classical plasmas in the presence of external perturbations.

In 2012, ref. [28] studied the generalized KP-MEW equation by using bifurcation theory of planar dynamical systems and obtained some exact traveling wave solutions such as solitary wave solution, periodic wave solutions and compactons. Recently, refs. [29]-[32] investigated nonlinear traveling waves in plasmas in the frameworks of KP, MKP and ZK equations respectively and obtained exact traveling wave solutions using bifurcation theory of planar dynamical systems. But, none of the work investigated nonlinear ion acoustic waves in e-p-i plasmas applying the bifurcation theory of planar dynamical systems through perturbative approach.

The bifurcation behavior of ion acoustic traveling waves in electron-positron-ion magnetoplasmas with superthermal electrons and positrons is studied in the framework of KP equation using bifurcation theory of planar dynamical systems. The solitary and the periodic wave solutions of the KP equation are derived. Considering external perturbation, the quasiperiodic behavior of the perturbed KP equation is studied in electron-positron-ion magnetoplasmas with $\kappa$ distributed electrons and positrons.
5.2 Model equations

We consider a plasma model whose constituents are cold ions, superthermal (kappa distributed) electrons and positrons in presence of an external static magnetic field \( M_0 = \hat{x}M_0 \) acting along the \( x \)-axis, where \( \hat{x} \) is the unit vector along the \( x \)-axis. The normalized continuity, momentum and Poisson’s equations are, respectively, given by:

\[
\begin{align*}
\frac{\partial n}{\partial t} + \nabla \cdot (n\tilde{U}) &= 0, \quad (5.1) \\
\frac{\partial \tilde{U}}{\partial t} + (\tilde{U} \cdot \nabla)\tilde{U} &= -\nabla \phi + \tilde{U} \times \hat{x}, \quad (5.2) \\
\nabla^2 \phi &= \alpha (n_e - n_p - n), \quad (5.3)
\end{align*}
\]

The normalized electron and positron densities are given by

\[
n_e = (1 + p)(1 - \frac{\phi}{\kappa-3/2})^{-(\kappa-1/2)} \quad \text{and} \quad n_p = p(1 + \frac{\delta \phi}{\kappa-3/2})^{-(\kappa-1/2)}. \]

The cause for considering \( \kappa \)-distributed electrons and positrons is discussed in detail in subsection 1.5.1, chapter 1 (see for details). Here \( p = \frac{n_p}{n_e} \), \( \alpha = \frac{r^2}{\lambda^2} \), \( r = \frac{C_s}{\Omega} \) is the ion gyroradius, \( \lambda = \sqrt{T_e/4\pi e^2n_0} \) is the electron Debye length, \( C_s = (T_e/m)^{1/2} \) is the ion acoustic velocity and \( \Omega = \frac{4\pi e^2 n_0}{mc} \) is the ion gyrofrequency, where \( c \) is the speed of the light and \( m \) is the mass of ions. \( \delta = \frac{T_e}{T_p} \) is the electron-to-positron temperature ratio and \( \phi \) is the electrostatic potential. \( n \) and \( \tilde{U} \) denote the density and the velocity of ions respectively. We assume that the wave is propagating in the \( xy \)-plane. Here \( n_{e0}, n_{p0}, \) and \( n_0 \) are respectively unperturbed densities of electrons, positrons and ions. The ion velocity \( \tilde{U} = (u,v,w) \) is normalized to ion acoustic speed \( C_s = \sqrt{T_e/m} \) and electrostatic potential \( \phi \) is normalized to \( T_e/e \), where \( e \) denotes the electron’s charge. Space variables and time are normalized to the electron Debye length \( \lambda = \sqrt{T_e/4\pi e^2n_0} \) and inverse of the ion gyrofrequency frequency \( \Omega = \sqrt{4\pi e^2n_0/m} \) respectively.

Equations (5.1)-(5.3) can be written in Cartesian coordinates as follows:
\[ \frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} + \frac{\partial (nv)}{\partial y} = 0, \quad (5.4) \]
\[ \frac{\partial u}{\partial t} + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y})u = -\frac{\partial \phi}{\partial x}, \quad (5.5) \]
\[ \frac{\partial v}{\partial t} + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y})v = -\frac{\partial \phi}{\partial y} + w, \quad (5.6) \]
\[ \frac{\partial w}{\partial t} + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y})w = -v, \quad (5.7) \]
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = \alpha \{(1 + p)(1 - \frac{\phi}{\kappa - 3/2})^{-(\kappa - 1/2)} - p(1 + \frac{\delta \phi}{\kappa - 3/2})^{-(\kappa - 1/2)} - n \}. \quad (5.8) \]

5.3 KP equation

We want to apply the RPT to derive the KP equation. According to the RPT, the independent variables are stretched as follows:

\[ Y = \epsilon^2 y, \quad (5.9) \]
\[ \eta = \epsilon (x - Vt), \quad (5.10) \]
\[ \tau = \epsilon^3 t, \quad (5.11) \]

where \( V \) is the phase velocity of ion acoustic wave along the \( x \)-axis in electron-positron-ion magnetoplasma with superthermal electrons and positrons. Here \( \epsilon \) is a small parameter which characterizes the strength of the nonlinearity. The dependent variables are expanded as follows:

\[ n = 1 + \epsilon^2 n_1 + \epsilon^4 n_2 + \ldots \quad (5.12) \]
\[ u = \epsilon^2 u_1 + \epsilon^4 u_2 + \ldots \quad (5.13) \]
\[ v = \epsilon^3 v_1 + \epsilon^5 v_2 + \ldots \quad (5.14) \]
\[ w = \epsilon^3 w_1 + \epsilon^5 w_2 + \ldots \quad (5.15) \]
\[ \phi = \epsilon^2 \phi_1 + \epsilon^4 \phi_2 + \ldots \quad (5.16) \]
Substituting the Eqs. (5.9)-(5.16) into the system of Eqs. (5.4)-(5.8) and equating the coefficient of lowest order of $\epsilon$, we obtain the dispersion relation as

$$V^2 = \frac{(\kappa - \frac{3}{2})}{(\kappa - \frac{1}{2})(1 + p + p\delta)}.$$  

(5.17)

Considering the coefficient of next order of $\epsilon$, one can obtain the KP equation as

$$\frac{\partial}{\partial \eta} [\frac{\partial \phi_1}{\partial \tau} + A\frac{\partial \phi_1}{\partial \eta} + B\frac{\partial^3 \phi_1}{\partial \eta^3}] + C\frac{\partial^2 \phi_1}{\partial Y^2} = 0,$$

(5.18)

where $A = \frac{1}{2V}[3 - \frac{(\kappa + \frac{5}{2})(1 + p + p\delta)}{(\kappa - \frac{1}{2})(1 + p + p\delta)^2}]$, $B = \frac{V}{2\alpha(1 + p + p\delta)}\frac{(\kappa - \frac{3}{2})}{(\kappa - \frac{1}{2})}$, and $C = \frac{V}{2}$.

### 5.4 Traveling wave system

In this section, we want to transform the KP equation (5.18) to the traveling wave system. We define a new variable $\chi$ as

$$\chi = \alpha_1(l\eta + mY - U\tau),$$

(5.19)

where $l$ and $m$ are the direction cosines of the angles made by the wave propagation with the $x$-axis and the $y$-axis respectively. Here $U$ is the speed of the traveling wave and $\alpha_1$ is a scalar quantity. Substituting $\psi(\chi) = \phi_1(\eta, Y, \tau)$ into Eq. (5.18) and then integrating twice, the KP equation (5.18) becomes

$$B\alpha^2_1l^4\frac{d^2\psi}{d\chi^2} + (Cm^2 - lU)\psi + \frac{Al^2}{2}\psi^2 = 0.$$  

(5.20)

Then Eq. (5.20) can be expressed to the following dynamical system:

$$\begin{align*}
\frac{d\phi}{d\chi} &= z, \\
\frac{dz}{d\chi} &= \frac{(U-Cm^2-4l^2\psi)\psi}{Bo\alpha^2_1l^4}. 
\end{align*}$$

(5.21)
The Eq.(5.21) is a planar Hamiltonian system with the Hamiltonian function:

\[
H(\psi, z) = \frac{z^2}{2} - \frac{1}{6B\alpha_1^4 l^4}(3(lU - Cm^2) - A^2\psi)\psi^2 = h, \text{ say.} \tag{5.22}
\]

The system (5.21) is a planar dynamical system with parameters \(\alpha, \alpha_1, \kappa, p, \delta, l, m\) and \(U\).

### 5.5 Bifurcations of phase portraits

In this section, we study the bifurcations of phase portraits of Eq.(5.21). When \(AB\alpha_1 l \neq 0\) and \(lU \neq C(1 - l^2)\), then there are two equilibrium points at \(E_0(\psi_0, 0)\) and \(E_1(\psi_1, 0)\), where \(\psi_0 = 0\) and \(\psi_1 = \frac{2(lU - C(1-l^2))}{Al^2}\). Let \(M(\psi_i, 0)\) be the coefficient matrix of the linearized system of Eq.(5.21) at an equilibrium point \(E_i(\psi_i, 0)\). Then we have

\[
J = \det M(\psi_i, 0) = \frac{(C(1-l^2) - lU)}{B\alpha_1^4 l^4} + \frac{A}{B\alpha_1^4 l^2} \psi_i. \tag{5.23}
\]

We consider \(0 < l < 1\) and \(\alpha_1 \neq 0\). Then we have the following cases:

**Case 1:** If \(2lU > V(1 - l^2), 3(\kappa - \frac{1}{2})(1 + p + p\delta)^2 > (\kappa + \frac{1}{2})(1 + p - p\delta^2), \alpha_1 l > 0, 0 < \kappa < \frac{1}{2}, \text{ and } \kappa > \frac{3}{2}\), then the system (5.21) has two equilibrium points at \(E_0(\psi_0, 0)\) and \(E_1(\psi_1, 0)\), where \(\psi_0 = 0\) and \(\psi_1 > 0\). Here \(E_0(\psi_0, 0)\) is a saddle point and \(E_1(\psi_1, 0)\) is a center. There is a homoclinic orbit to \(E_0(\psi_0, 0)\) surrounding the center \(E_1(\psi_1, 0)\) (see figure 5.1).

**Case 2:** If \(2lU < V(1 - l^2), 3(\kappa - \frac{1}{2})(1 + p + p\delta)^2 > (\kappa + \frac{1}{2})(1 + p - p\delta^2), \alpha_1 l > 0, 0 < \kappa < \frac{1}{2}, \text{ and } \kappa > \frac{3}{2}\), then the system (5.21) has two equilibrium points at \(E_0(\psi_0, 0)\) and \(E_1(\psi_1, 0)\), where \(\psi_0 = 0\) and \(\psi_1 < 0\). Here \(E_0(\psi_0, 0)\) is a center and \(E_1(\psi_1, 0)\) is a saddle point. There is a homoclinic orbit to \(E_1(\psi_1, 0)\) surrounding the center \(E_0(\psi_0, 0)\) (see figure 5.2).
By the theory of planar dynamical systems ([33]-[34] and applying the above analysis, different phase portraits of Eq.(5.21) for some special values of the parameters are shown in figures 5.1-5.2.

Figure 5.1: Phase portrait of Eq.(5.21) for $\alpha = 0.8, \alpha_1 = 1, \kappa = 4, l = 0.4, p = 0.8, \delta = 1.6$ and $U = 0.6$.

Figure 5.2: Phase portrait of Eq.(5.21) for $\alpha = 0.8, \alpha_1 = 1, \kappa = 0.1, l = 0.4, p = 0.8, \delta = 1.6$ and $U = 0.2$. 
5.6 Exact traveling wave solutions

In this section, with the help of the planar dynamical system Eq. (5.21) and the Hamiltonian function Eq. (5.22) with $h = 0$, we present the solitary wave solution and the periodic traveling wave solution of Eq. (5.18) specifying the constraints among parameters for existence of such solutions.

(1) If $2lU > V(1-l^2), 3(\kappa - \frac{1}{2})(1+p+p\delta)^2 > (\kappa + \frac{1}{2})(1+p-p\delta^2), \alpha \alpha_1 l > 0, 0 < \kappa < \frac{1}{2}$, and $\kappa > \frac{3}{2}$, (see Figures 5.1 and 5.3), the KP equation (5.18) has the solitary wave solution given by

$$\phi_1 = \frac{3(lU - C(1-l^2))}{A l^2} \text{sech}^2 \left( \frac{1}{2\alpha_1 l^2} \sqrt{\frac{lU - C(1-l^2)}{B}} \chi \right). \quad (5.24)$$

The derivation of the solution (5.24) is same as the solution (2.36) in subsection 2.2.5, chapter 2 (see for details).

(2) If $2lU < V(1-l^2), 3(\kappa - \frac{1}{2})(1+p+p\delta)^2 > (\kappa + \frac{1}{2})(1+p-p\delta^2), \alpha \alpha_1 l > 0, 0 < \kappa < \frac{1}{2}$, and $\kappa > \frac{3}{2}$, (see Figures 5.2 and 5.4), the KP equation (5.18) has the periodic traveling wave solution given by

$$\phi_1 = \frac{3(lU - C(1-l^2))}{A l^2} \csc^2 \left( \frac{1}{2\alpha_1 l^2} \sqrt{\frac{lU - C(1-l^2)}{B}} \chi \right). \quad (5.25)$$

The derivation of the solution (5.25) is same as the solution (2.35) in subsection 2.2.5, chapter 2 (see for details).

Using symbolic computations, we obtain two graphs of these exact solitary wave solutions and periodic traveling wave solutions of Eq.(5.18) when the parameters satisfy some conditions. These are shown in figures 5.3-54.
Figure 5.3: Variation of the solitary wave profiles of Eq.(5.18) for different values of $\kappa$ with $\alpha = 0.8, \alpha_1 = 1, l = 0.4, p = 0.8, \delta = 1.6$ and $U = 0.6$.

Figure 5.4: Variation of the Periodic traveling wave profiles of Eq.(5.18) for different values of $\kappa$ with $\alpha = 0.8, \alpha_1 = 1, l = 0.4, p = 0.8, \delta = 1.6$ and $U = 0.2$.

Solitary wave profiles for different values of $\kappa(2, 3, 4)$ are plotted in figure 5.3. The values of other parameters are given in the figure caption. It is clear that when $\kappa$ decreases, the amplitude of the solitary wave increases but its width decreases and it becomes more spiky. Thus, one can conclude that the ion acoustic solitary waves may
become smoother as the electrons and positrons evolve nearby from their Maxwell-Boltzmann equilibrium.

In figure 5.4, we have presented variation of the periodic wave profiles for different values of \( \kappa (0.1, 0.2, 0.3) \) with fixed values of the other parameters. It is clear that when \( \kappa \) increases, amplitude of the periodic wave decreases and width of the periodic wave increases. Thus, one can conclude that the ion acoustic periodic waves may become less smooth as the electrons and positrons evolve far away from their Maxwell-Boltzmann equilibrium.

### 5.7 Quasiperiodicity

In this section, we will discuss the quasiperiodic behavior of the following perturbed system:

\[
\begin{align*}
\frac{d\psi}{d\chi} &= \frac{z}{B_\alpha}, \\
\frac{dz}{d\chi} &= \frac{(U - Cm^2 - \frac{4\pi^2}{B_\alpha^2} \psi)\psi}{B_\alpha} + f_0 \cos(\omega \chi),
\end{align*}
\]

(5.26)

where \( f_0 \cos(\omega \chi) \) is the external forcing term, \( f_0 \) is strength of the external force and \( \omega \) is the frequency. The difference between the system (5.21) and the system (5.32) is that only external periodic perturbation is added with the system (5.32).

In figures 5.5-5.8, we have presented phase portraits of the perturbed system (5.32) for different values of \( \kappa (0.1, 0.2, 3, 4) \) with fixed values of the other parameters \( \alpha = 0.8, \alpha_1 = 1, l = 0.4, U = 0.2, p = 0.8, f_0 = 1, \omega = 1 \) and \( \delta = 1.6 \). A quasi periodic motion of the system (5.32) is found with incommensurable periodic motions and the trajectory in the phase space winds around torus filling its surface densely.
Figure 5.5: Phase portrait of Eq.(5.32) for $\alpha = 0.8, \alpha_1 = 1, \kappa = 0.1, l = 0.4, U = 0.2, p = 0.8, f_0 = 1, \omega = 1$ and $\delta = 1.6$.

Figure 5.6: Phase portrait of Eq.(5.32) for $\kappa = 0.2$ and other parameters are same as figure 5.5.

Figure 5.7: Phase portrait of Eq.(5.32) for $\kappa = 3$ and other parameters are same as figure 5.5.

Figure 5.8: Phase portrait of Eq.(5.32) for $\kappa = 4$ and other parameters are same as figure 5.5.

In figures 5.9-5.12, we have plotted $z$ vs. $\chi$ for the perturbed system (5.32) for different values of $\kappa(0.1, 0.2, 3, 4)$ with fixed values of the other parameters which are $\alpha = 0.8, \alpha_1 = 1, l = 0.4, U = 0.2, p = 0.8, f_0 = 1, \omega = 1$ and $\delta = 1.6$. It is easily seen that stable oscillatory behavior is possible in the system (5.32) for different values of $\kappa$. The presence of slow and fast frequency components are visible in time evolution of the state variable.
From figures 5.5-5.12, it is clear that the perturbed system (5.32) has the quasi-periodic behavior but not chaotic in presence of the external force. Ref. [26] studied the solitonic, the quasi periodic and the periodic patterns of nonlinear electron acoustic waves in quantum plasmas. Recently, ref. [27] investigated dynamic behavior of ion acoustic waves in dense quantum magnetoplasmas in presence of the external perturbation. Thus the feature "quasi-periodicity" of our work has been supported by the works reported by previous authors [26]-[27].
5.8 Conclusions

In this chapter, we have derived the KP equation in electron-positron-ion magnetoplasma with kappa distributed electrons and positrons. Using the bifurcation theory of planar dynamical systems to the KP equation, we have presented the existence of the solitary wave solutions and the periodic traveling wave solutions. Two exact solutions of these waves are obtained depending on the system parameters $\alpha, \alpha_1, \kappa, p, \delta, l,$ and $U$. Considering an external perturbation, the quasiperiodic behavior of ion acoustic waves has been studied. The spectral index $\kappa$ plays a significantly in the propagation of the multi-soliton of the KP equation and the quasiperiodicity of the perturbed KP equation.
Bibliography


