Chapter 3

OSCILLATION OF NONLINEAR NEUTRAL DIFFERENCE EQUATIONS OF THIRD ORDER

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3.1 Introduction

In this chapter, some sufficient conditions for oscillation of all solutions of certain difference equations are obtained. Examples are given to illustrate the results.

We are concerned with oscillation behavior of the solutions of a third order nonlinear neutral difference equations of the form

$$\Delta^2 \left( \frac{q_n}{a_n} \Delta(x_n + c_n x_{n-\sigma}) \right) + p_n \Delta x_n + q_n f(x_{n+1}) = 0, \quad n = 0, 1, 2 \ldots, \quad (3.1.1)$$

where the following conditions are assumed to be hold.

(H1) \{a_n\}, \{p_n\}, \{q_n\} and \{c_n\} are real positive sequences and \(q_n \neq 0\) for infinitely many values of \(n\).
(H2) \( f : R \to R \) is continuous and \( xf(x) > 0 \) for all \( x \neq 0 \).

(H3) there exists a real valued function \( g \) such that
\[
f(u_n) - f(v_n) = g(u_n, v_n)[(u_n + c_n u_{n-\sigma}) - (v_n + c_n v_{n-\sigma})], \quad \text{for all } u_n \neq 0,
\]
\[
v_n \neq 0, \ n > \sigma > 0 \text{ and } g(u_n, v_n) \geq L > 0.
\]

(H4) \( \phi : R \to R \) is continuous for all \( x \neq 0 \), \( \phi(x) > 0 \).

(H5) \[
\sum_{n=M}^{\infty} (n+1)p_n^2 < \infty.
\]

(H6) \[
\sum_{n=M}^{\infty} \frac{q_n^2}{a_n^2} < \infty.
\]

(H7) \[
\sum_{n=M}^{\infty} (n+1)q_n = \infty.
\]

(H8) \[
\sum_{n=M}^{\infty} \frac{a_n}{nq_n} = \infty.
\]

(H9) \[
\sum_{n=M}^{\infty} \frac{1}{sb_s} = \infty.
\]

3.2 Results Related to Oscillation Behavior

Theorem 3.2.1. In addition to (H1), (H2) and (H3), assume that (H5), (H6), (H7), (H8) and (H9) hold and let \( z_n = x_n + c_n x_{n-\sigma} \). Then, every solution of (3.1.1) is oscillatory.

Proof. Suppose the contrary and assume that \( \{x_n\} \) be a non-oscillatory solution of (3.1.1), such that \( x_n > 0 (or x_n < 0) \) for all \( n \geq M - 1 \), \( M > 0 \) is an integer and let \( b_n = \frac{q_n}{a_n} \). Equation (3.1.1) implies
\[
\Delta(b_{n+1}\Delta z_{n+1}) - \Delta(b_n\Delta z_n) + p_n\Delta x_n + q_n f(x_{n+1}) = 0 \quad (3.2.1)
\]
Multiplying (3.2.1) by \(\frac{n+1}{f(z_{n+1})}\) and summing from \(M\) to \((n-1)\), we obtain

\[
\sum_{s=M}^{n-1} \left( \frac{s + 1}{f(z_{s+1})} \right) \Delta (b_{s+1} \Delta z_{s+1}) - \sum_{s=M}^{n-1} \left( \frac{s + 1}{f(z_{s+1})} \right) \Delta (b_s \Delta z_s) + \sum_{s=M}^{n-1} \left( \frac{s + 1}{f(x_{s+1})} \right) p_s \Delta x_s + \sum_{s=M}^{n-1} (s + 1) q_s = 0. \tag{3.2.2}
\]

But

\[
\sum_{s=M}^{n-1} \left( \frac{s + 1}{f(x_{s+1})} \right) \Delta (b_{s+1} \Delta z_{s+1}) = \left( \left( \frac{s + 1}{f(x_{s+1})} \right) (b_{s+1} \Delta z_{s+1}) \right)_{s=M}^{n} - \sum_{s=M}^{n-1} \Delta \left( \frac{s + 1}{f(x_{s+1})} \right) (b_{s+2} \Delta z_{s+2})
\]

\[
= \frac{(n + 1) b_{n+1} \Delta x_{n+1}}{f(x_{n+1})} - \frac{(M + 1) b_{M+1} \Delta z_{M+1}}{f(x_{M+1})} - \sum_{s=M}^{n-1} \frac{b_{s+2} \Delta z_{s+2}}{f(x_{s+2})} + \sum_{s=M}^{n-1} \frac{(s + 1) b_{s+2} g(x_{s+2}, x_{s+1}) \Delta z_{s+1} \Delta z_{s+2}}{f(x_{s+1}) f(x_{s+2})}. \tag{3.2.3}
\]

Also,

\[
\sum_{s=M}^{n-1} \left( \frac{s + 1}{f(x_{s+1})} \right) \Delta (b_s \Delta z_s) = \left( \left( \frac{s + 1}{f(x_{s+1})} \right) (b_s \Delta z_s) \right)_{s=M}^{n} - \sum_{s=M}^{n-1} \Delta \left( \frac{s + 1}{f(x_{s+1})} \right) (b_{s+1} \Delta z_{s+1})
\]

\[
= \frac{(n + 1) b_n \Delta z_n}{f(x_{n+1})} - \frac{(M + 1) b_M \Delta z_M}{f(x_{M+1})} - \sum_{s=M}^{n-1} \frac{b_{s+1} \Delta z_{s+1}}{f(x_{s+2})} + \sum_{s=M}^{n-1} \frac{(s + 1) b_{s+1} g(x_{s+2}, x_{s+1}) (\Delta z_{s+1})^2}{f(x_{s+1}) f(x_{s+2})}. \tag{3.2.4}
\]
Substituting (3.2.3) and (3.2.4) in (3.2.2), we have

$$\left(\frac{n+1}{f(x_{n+1})} - \frac{n+1}{f(x_n)}\right) - \sum_{s=M}^{n-1} \left(\frac{b_{s+2} \Delta z_{s+2}}{f(x_{s+2})} - \frac{b_{s+1} \Delta z_{s+1}}{f(x_{s+2})}\right)$$

$$+ \sum_{s=M}^{n-1} \left(\frac{(s+1)b_{s+2}g(x_{s+2}, x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})}\right)$$

$$- \sum_{s=M}^{n-1} \left(\frac{(s+1)b_{s+1}g(x_{s+2}, x_{s+1})(\Delta z_{s+1})^2}{f(x_{s+1})f(x_{s+2})}\right)$$

$$+ \sum_{s=M}^{n-1} \left(\frac{s+1}{f(x_{s+1})} p_s \Delta x_s + \sum_{s=M}^{n-1} (s+1)q_s\right)$$

$$= \left(\frac{(M+1)\Delta z_{M+1}}{f(x_{M+1})} - \frac{(M+1)\Delta z_M}{f(x_{M+1})}\right).$$

(3.2.5)

Using Schwarz’s inequality, we have

$$\sum_{s=M}^{n-1} \left(\frac{b_{s+2} \Delta z_{s+2}}{f(x_{s+2})}\right)^2 \leq \left(\sum_{s=M}^{n-1} b_{s+2}^2 \right)^{\frac{1}{2}} \left(\sum_{s=M}^{n-1} \frac{(\Delta z_{s+2})^2}{f^2(x_{s+2})}\right)^{\frac{1}{2}},$$

(3.2.6)

$$\sum_{s=M}^{n-1} \left(\frac{b_{s+1} \Delta z_{s+1}}{f(x_{s+2})}\right)^2 \leq \left(\sum_{s=M}^{n-1} b_{s+1}^2 \right)^{\frac{1}{2}} \left(\sum_{s=M}^{n-1} \frac{(\Delta z_{s+1})^2}{f^2(x_{s+2})}\right)^{\frac{1}{2}},$$

(3.2.7)

$$\sum_{s=M}^{n-1} \left(\frac{(s+1)b_{s+2}g(x_{s+2}, x_{s+1})\Delta z_{s+1}\Delta x_{s+2}}{f(x_{s+1})f(x_{s+2})}\right)$$

$$\leq \left(\sum_{s=M}^{n-1} b_{s+2}^2 \right)^{\frac{1}{2}} \left(\sum_{s=M}^{n-1} \frac{(s+1)^2 g^2(x_{s+2}, x_{s+1})(\Delta z_{s+1})^2(\Delta z_{s+2})^2}{f^2(x_{s+1})f^2(x_{s+2})}\right)^{\frac{1}{2}},$$

(3.2.8)

$$\sum_{s=M}^{n-1} \left(\frac{(s+1)b_{s+1}g(x_{s+2}, x_{s+1})(\Delta z_{s+1})^2}{f(x_{s+1})f(x_{s+2})}\right)$$

$$\leq \left(\sum_{s=M}^{n-1} b_{s+1}^2 \right)^{\frac{1}{2}} \left(\sum_{s=M}^{n-1} \frac{(s+1)^2 g^2(x_{s+2}, x_{s+1})(\Delta z_{s+1})^4}{f^2(x_{s+1})f^2(x_{s+2})}\right)^{\frac{1}{2}},$$

(3.2.9)
\[
\sum_{s=M}^{n-1} \frac{(s+1)p_s \Delta x_s}{f(x_{s+1})} \leq \left( \sum_{s=M}^{n-1} (s+1)p_s^2 \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{(s+1)(\Delta x_s)^2}{f^2(x_{s+1})} \right)^{\frac{1}{2}}. 
\] 

(3.2.10)

In view of (3.2.6), (3.2.7), (3.2.8), (3.2.9) and (3.2.10), the summation in (3.2.5) is bounded and so, we have

\[
\left( \frac{(n+1)b_{n+1} \Delta z_{n+1}}{f(x_{n+1})} - \frac{(n+1)b_n \Delta z_n}{f(x_{n+1})} \right) - \left( \sum_{s=M}^{n-1} b_{s+1}^2 \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{(\Delta z_{s+1})^2}{f^2(x_{s+2})} \right)^{\frac{1}{2}}
\]

\[
+ \left( \sum_{s=M}^{n-1} b_{s+1}^2 \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{b_{s+1}^2}{f^2(x_{s+2})} \right) + \left( \sum_{s=M}^{n-1} (s+1)p_s^2 \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{(s+1)(\Delta x_s)^2}{f^2(x_{s+1})} \right)^{\frac{1}{2}}
\]

\[
- \left( \sum_{s=M}^{n-1} b_{s+1}^2 \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{(s+1)^2 g^2(x_{s+2}, x_{s+1})(\Delta z_{s+1})^2}{f^2(x_{s+1})f^2(x_{s+2})} \right)^{\frac{1}{2}}
\]

\[
+ \left( \sum_{s=M}^{n-1} b_{s+2}^2 \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{b_{s+2}^2}{f^2(x_{s+2})} \right) \left( \frac{(M+1)b_{M+1} \Delta z_{M+1}}{f(x_{M+1})} - \frac{(M+1)b_M \Delta z_M}{f(x_{M+1})} \right) - \sum_{s=M}^{n-1} (s+1)q_s. 
\] 

(3.3.11)

In view of (H5), (H6), (H7) and (H8), we get from (3.2.11) that

\[
\frac{(n+1)(b_{n+1} \Delta z_{n+1} - b_n \Delta z_n)}{f(x_{n+1})} \to -\infty, \text{ as } n \to \infty
\]

That is,

\[
\frac{(n+1)\Delta (b_n \Delta z_n)}{f(x_{n+1})} \to -\infty, \text{ as } n \to \infty.
\]

Hence there exists \( M_1 \geq M \) such that \( \Delta (a_n \Delta z_n) < 0 \) for \( n \geq M \)
which implies \( \Delta (b_n \Delta z_n) < -k, k > 0 \).

Summing the last inequality from \( m \geq M_1 \) to \( (n - 1) \), we obtain

\[
(b_n \Delta z_n)^{n}_{s=m} < (-k)(n - m),
\]

which implies \( b_n \Delta z_n < -k(n - m) + b_m \Delta z_m \). Therefore \( b_n \Delta z_n \to -\infty \), as

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\[ n \to \infty. \text{ Hence there exists} \]
\[ M_2 \geq M_1 \text{ such that } \Delta z_n < 0, \text{ for } n \geq M_2. \]  \hfill (3.2.12)

Rewriting (3.2.5), we have
\[
\frac{(n+1)b_{n+1}\Delta z_{n+1}}{f(x_{n+1})} + \sum_{s=M_2}^{n-1} \frac{(s+1)b_{s+2}g(x_{s+2}, x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})}
\]
\[
= \frac{(n+1)b_n\Delta z_n}{f(x_{n+1})} + \frac{(M+1)b_{M+1}\Delta z_{M+1}}{f(x_{M+1})} - \frac{(M+1)b_M\Delta z_M}{f(x_{M+1})}
\]
\[- \sum_{s=M}^{M-1} \frac{(s+1)q_s}{f(x_{s+1})} - \sum_{s=M}^{n-1} \frac{(s+1)b_{s+1}g(x_{s+2}, x_{s+1})(\Delta z_{s+1})^2}{f(x_{s+1})f(x_{s+2})}
\]
\[- \sum_{s=M}^{M-1} \frac{(s+1)b_{s+1}g(x_{s+2}, x_{s+1})(\Delta z_{s+1})^2}{f(x_{s+1})f(x_{s+2})}
\]
\[- \sum_{s=M}^{M-1} \frac{(s+1)p_s\Delta x_s}{f(x_{s+1})} - \sum_{s=M}^{n-1} \frac{(s+1)p_s\Delta x_s}{f(x_{s+1})}. \]  \hfill (3.2.13)

From (H1), (H7), (H8), (3.2.12) and (3.2.13), there exists an integer \( M_3 \geq M_2 \), such that
\[
\frac{(n+1)b_{n+1}\Delta z_{n+1}}{f(x_{n+1})} + \sum_{s=M_2}^{n-1} \frac{(s+1)b_{s+2}g(x_{s+2}, x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})} \leq -k, k \geq M_3,
\]
where \( k \) is a positive constant. Therefore,
\[
\frac{-(n+1)b_{n+1}\Delta z_{n+1}}{f(x_{n+1})} - \sum_{s=M_2}^{n-1} \frac{(s+1)b_{s+2}g(x_{s+2}, x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})} \geq k. \]  \hfill (3.2.14)

Now let \( u_{n+1} = -(n+1)\Delta z_{n+1} \), (3.2.14) becomes
\[
u_{n+1} \geq k + \sum_{s=M_3}^{n-1} \frac{(s+1)b_{s+2}g(x_{s+2}, x_{s+1})\Delta z_{s+1}\Delta z_{s+2}}{f(x_{s+1})f(x_{s+2})}, \quad n \geq M_3,
\]
\[
u_{n+1} \geq k + \frac{f(x_{n+1})}{b_{n+1}} + \sum_{s=M_3}^{n-1} \frac{b_{s+2}f(x_{n+1})g(x_{s+2}, x_{s+1})(-\Delta z_{s+2}u_{s+1})}{b_{n+1}f(x_{s+1})f(x_{s+2})}. \]  \hfill (3.2.15)

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Also, let \( v_{n+1} = k \frac{f(x_{n+1})}{b_{n+1}} + \sum_{s=M_3}^{n-1} \frac{b_{s+2} f(x_{s+1}) g(x_{s+2}, x_{s+1}) (-\Delta z_{s+2}) v_{s+1}}{b_{n+1} f(x_{s+1}) f(x_{s+2})} \).

(3.2.16)

Using lemma 2.2.1, we have from (3.2.15) and (3.2.16),

\[ u_{n+1} \geq v_{n+1} \]  
(3.2.17)

Therefore, equation (3.2.15) implies

\[ v_{n+1} = \frac{f(x_{n+1})}{b_{n+1}} \left( k + \sum_{s=M_3}^{n-1} \frac{b_{s+2} g(x_{s+2}, x_{s+1}) (-\Delta z_{s+2}) v_{s+1}}{f(x_{s+1}) f(x_{s+2})} \right) \]

This implies

\[ v_{n+1} \geq \frac{kf(x_{M_3})}{b_{n+1}}, \text{ for } n \geq M_3. \]  
(3.2.18)

From (3.2.17) and (3.2.18), we have

\[ -(n+1)\Delta z_{n+1} \geq \frac{kf(x_{M_3})}{b_{n+1}}. \]

That is, \( \Delta z_{n+1} \leq \frac{-kf(x_{M_3})}{(n+1)b_{n+1}}. \)  
(3.2.19)

Summing (3.2.19) from \( M_3 \) to \( (n-1) \), we have

\[ \sum_{s=M_3}^{n-1} \Delta z_{s+1} \leq -kf(x_{M_3}) \sum_{s=M_3}^{n-1} \frac{1}{(s+1)b_{s+1}}. \]

That is, \( (z_{s+1})_{s=M_3}^{n} \leq -kf(x_{M_3}) \sum_{s=M_3}^{n-1} \frac{1}{(s+1)b_{s+1}}, \)

which implies \( z_{n+1} \leq z_{M_3+1} - kf(x_{M_3}) \sum_{s=M_3}^{n-1} \frac{1}{(s+1)b_{s+1}}. \)

In view of (H9), \( z_n = (x_n + c_n x_{n-\sigma}) \leq 0, \) for sufficiently large \( n \), which is a contradiction to the fact that \( x_n \) is eventually positive. The proof is similar for the case when \( x_n \) is eventually negative. Hence the proof is complete. \( \Box \)
Corollary 3.2.1. In addition to (H1), (H2), (H3) and (H4), assume that (H5),
(H6), (H7), (H8) and (H9) hold, \( \phi : R \to R \) is continuous for all \( x \neq 0 \),
\( \phi(x) > 0 \) and let \( z_n = x_n + c_n x_{n-\sigma} \). Then, every solution of the equation
\[ \Delta^2 \left( \frac{q_n}{a_n} \phi(x_n) \Delta z_n \right) + p_n \Delta x_n + q_n f(x_n) = 0 \]
is oscillatory.

Corollary 3.2.2. In addition to (H1), (H2) and (H3), assume that (H6), (H7),
(H8) and (H9) hold and let \( z_n = x_n + c_n x_{n-\sigma} \). Then, every solution of the
equation \[ \Delta^2 \left( \frac{q_n}{a_n} \Delta z_n \right) + q_n f(x_{n+1}) = 0 \]
is oscillatory.

Corollary 3.2.3. In addition to (H1), (H2), (H4) and (H4), assume that (H6),
(H7), (H8) and (H9) hold, \( \phi : R \to R \) is continuous for all \( x \neq 0 \), \( \phi(x) > 0 \) and
let \( z_n = x_n + c_n x_{n-\sigma} \). Then, every solution of the equation \[ \Delta^2 \left( \frac{q_n}{a_n} \phi(x_n) \Delta z_n \right) + q_n f(x_n) = 0 \]
is oscillatory.

Proof. Proofs for Corollary 3.2.1, Corollary 3.2.2 and Corollary 3.2.3 are similar
to the proof of Theorem 3.2.1 and hence the details are omitted. \( \Box \)

3.3 Examples

Example 3.3.1. Consider the difference equation
\[ \Delta^2 \left( \frac{n}{n+1} \Delta(x_n + 5x_{n-\sigma}) \right) + \frac{9n^2 + 18n + 5}{2n^2(n+1)(n+2)} \Delta x_n + \frac{1}{n(n+1)} x_{n+1} = 0. \]

(E.3.1)

Here \( a_n = \frac{1}{n^2}, \ p_n = \frac{9n^2 + 18n + 5}{2n^2(n+1)(n+2)}, \ q_n = \frac{1}{n(n+1)}, \ b_n = \frac{q_n}{a_n} = \frac{n}{n+1}, \)
\[ \sum_{n=2}^{\infty} \frac{1}{n b_n} = \sum_{n=2}^{\infty} \frac{n+1}{n^2} = \infty, \ \sum_{n=2}^{\infty} \frac{b_n^2}{n^2} = \sum_{n=2}^{\infty} \frac{n^2}{(n+1)^2} < \infty, \]
∞ \sum_{n=2}^{\infty} (n + 1)p_n^2 = \sum_{n=2}^{\infty} \frac{(9n^2 + 18n + 5)^2}{4n^4(n + 1)(n + 2)^2} < \infty \text{ and }
∞ \sum_{n=2}^{\infty} (n + 1)q_n = \sum_{n=2}^{\infty} \frac{1}{n} = \infty. \text{ All the conditions of theorem 3.2.1 are satisfied.}

Hence equation (E.3.1) is oscillatory.

Example 3.3.2. Consider the difference equation

$$\Delta^2 \left( \frac{n+1}{n+2} \Delta(x_n + 3x_{n-\sigma}) \right) + \frac{1}{n^3} \sqrt{\frac{n}{n+1}} \Delta x_n + \frac{1}{(n+1)(n+2)} x_{n+1}^3 = 0.$$  

(E.3.2)

Here \( a_n = \frac{1}{(n+1)^2} \), \( p_n = \frac{1}{n^3} \sqrt{\frac{n}{n+1}} \), \( q_n = \frac{1}{(n+1)(n+2)} \), \( b_n = \frac{n+1}{n+2} \),

\[ \sum_{n=2}^{\infty} \frac{1}{n} b_n = \sum_{n=2}^{\infty} \frac{n+2}{n(n+1)} = \infty, \quad \sum_{n=2}^{\infty} b_n^2 = \sum_{n=2}^{\infty} \left( \frac{n+1}{n+2} \right)^2 < \infty, \]

\[ \sum_{n=2}^{\infty} (n+1)p_n^2 = \sum_{n=2}^{\infty} \frac{1}{n^5} < \infty \text{ and } \]

\[ \sum_{n=2}^{\infty} (n+1)q_n = \sum_{n=2}^{\infty} \frac{1}{n+2} = \infty. \text{ All the conditions of theorem 3.2.1 are satisfied.} \]

Hence equation (E.3.2) is oscillatory.

Example 3.3.3. Consider the difference equation

$$\Delta^2 \left( \frac{1}{n(2n+1)} 4x_n^2 \Delta(x_n + \frac{n}{n-2} x_{n-2}) \right) + \frac{2}{n^2} \Delta x_n + \frac{2(4n^3 + 4n^2 + 2n + 1)}{n^3} x_n = 0,$$  

\( n \geq 3. \)  

(E.3.3)

Here \( a_n = \frac{2(2n+1)(4n^3 + 4n^2 + 2n + 1)}{n^2}, \) \( p_n = \frac{2}{n^2}, \) \( q_n = \frac{2(4n^3 + 4n^2 + 2n + 1)}{n^3}, \)

\[ b_n = \frac{1}{n(2n+1)}, \quad c_n = \frac{n}{n-2}, \quad \sum_{n=3}^{\infty} \frac{1}{n} b_n = \sum_{n=3}^{\infty} (2n + 1) = \infty, \]

\[ \sum_{n=3}^{\infty} b_n^2 = \sum_{n=3}^{\infty} \frac{1}{n^2(2n+1)^2} < \infty, \quad \sum_{n=3}^{\infty} (n+1)p_n^2 = \sum_{n=3}^{\infty} \frac{4(n+1)}{n^4} < \infty \text{ and } \]

\[ \sum_{n=3}^{\infty} (n+1)q_n = \sum_{n=3}^{\infty} \frac{2(n+1)(4n^3 + 4n^2 + 2n + 1)}{n^3} = \infty. \text{ All the conditions of} \]

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corollary 3.2.1 are satisfied. Hence equation (E.3.3) is oscillatory. One such solution of equation (E.3.3) is \( x_n = \frac{n(-1)^n}{2} \).

**Example 3.3.4.** Consider the difference equation

\[
\Delta^2 \left( \frac{1}{n(2n+1)} 4x_n^2 \Delta(x_n + \frac{n}{n-2} x_{n-2}) \right) + \frac{32(n+1)}{n^3} x_n^3 = 0, \quad n \geq 3. \tag{E.3.4}
\]

Here \( a_n = \frac{32(n+1)(2n+1)}{n^2} \), \( q_n = \frac{32(n+1)}{n^3} \), \( b_n = \frac{1}{n(2n+1)} \), \( c_n = \frac{n}{n-2} \),

\[
\sum_{n=3}^{\infty} \frac{1}{nb_n} = \sum_{n=3}^{\infty} (2n+1) = \infty, \quad \sum_{n=3}^{\infty} b_n^2 = \sum_{n=3}^{\infty} \frac{1}{n^2(2n+1)^2} < \infty \quad \text{and}
\]

\[
\sum_{n=3}^{\infty} (n+1)q_n = \sum_{n=3}^{\infty} \frac{32(n+1)^2}{n^3} = \infty. \quad \text{All the conditions of corollary 3.2.3 are satisfied. Hence equation (E.3.4) is oscillatory. One such solution of equation (E.3.3) is } x_n = \frac{n(-1)^n}{2}.\]