Chapter 8

OSCILLATORY BEHAVIOR OF FOURTH ORDER NEUTRAL DELAY DIFFERENCE EQUATIONS

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8.1 Introduction

In this chapter, we are concerned with the oscillatory properties of solutions of fourth order neutral delay difference equation of the form

\[ \Delta(c_n \Delta^2(a_n \Delta(y_n + b_n y_{n-\tau}))) + q_n f(y_{n-\sigma}) = 0, \quad n \in \mathbb{N}, \]  

(8.1.1)

where the following conditions are assumed to be hold.

(H1) \( \tau, \sigma \) are non-negative constants.

(H2) \( c_n > 0, a_n > 0 \) and \( \sum_{n=n_0}^{\infty} \frac{1}{a_n} = \sum_{n=n_0}^{\infty} \frac{1}{b_n} = \sum_{n=n_0}^{\infty} \frac{1}{c_n} = \infty \), for \( n \geq n_0 \).

(H3) \( b_n \geq 0, q_n \neq 0 \) for infinitely many values of \( n \).
(H4) \( f : R \to R \) is continuous and \( xf(x) > 0 \) for all \( x \neq 0 \) and \( \frac{f(x)}{x} \geq K > 0 \).

(H5) \( \phi : N_0 \to R \) is continuous for all \( x \neq 0 \in N_0 \).

A non trivial solution \( \{y_n\} \) of equation (8.1.1) is said to be oscillatory if for any \( n_1 \geq n_0 \) there exists \( n \geq n_1 \) such that \( y_n y_{n+1} \leq 0 \), otherwise, the solution is said to be non-oscillatory.

8.2 Basic Lemmas

Lemma 8.2.1. If \( y_n \) is an eventually positive solution of equation (8.1.1) and \( z_n = y_n + b_n y_{n-\tau} \) then for sufficiently large \( n \), there are only two possible cases:

(I) \( z_n > 0 ; \Delta z_n > 0 ; \Delta(a_n \Delta z_n) > 0 ; \Delta^2(a_n \Delta z_n) > 0 \), (or)

(II) \( z_n > 0 ; \Delta z_n < 0 ; \Delta(a_n \Delta z_n) > 0 ; \Delta^2(a_n \Delta z_n) > 0 \)

Proof. Let \( y_n \) be an eventually positive solution of equation (8.1.1), then there exists \( n_1 \geq n_0 \) such that \( y_{n-\tau} > 0 \) and \( y_{n-\sigma} > 0 \) for \( n \geq n_1 \).

From the definition of \( z_n \), it is clear that \( z_n > 0 \) and \( \Delta(c_n \Delta^2(a_n \Delta z_n)) \leq 0 \) for \( n \geq n_1 \), thus \( z_n, \Delta z_n \) and \( \Delta^2(a_n \Delta z_n) \) are eventually of one sign.

We claim that

\[
\Delta^2(a_n \Delta z_n) > 0, \text{ for } n \geq n_2, \text{ for } n_2 \geq n_1. \tag{8.2.1}
\]

Suppose \( \Delta^2(a_n \Delta z_n) \leq 0 \) for all large \( n \).

Since \( q_n > 0 \) and \( c_n > 0 \), it is clear that there is an integer \( n_3 \geq n_2 \), we have

\[
c_n \Delta^2(a_n \Delta z_n) \leq c_{n_3} \Delta^2(a_{n_3} \Delta z_{n_3}) < 0. \tag{8.2.2}
\]
Summing the equation (8.2.2) from $n_3$ to $(n - 1)$, we have

$$\Delta(a_n \Delta z_n) - \Delta(a_{n_3} \Delta z_{n_3}) < c_{n_3} \Delta^2(a_{n_3} \Delta z_{n_3}) \sum_{s=n_3}^{n-1} \frac{1}{c_s}.$$  

In view of (H2), we see that $\Delta(a_n \Delta z_n) \to -\infty$ as $n \to \infty$ thus, there exists $n_4 \geq n_3$ such that

$$\Delta(a_n \Delta z_n) \leq \Delta(a_{n_4} \Delta z_{n_4}) < 0 \text{ for } n \geq n_4.$$  

Summing from $n_4$ to $(n - 1)$, we obtain

$$a_n \Delta z_n \to -\infty \text{ as } n \to \infty \quad (8.2.3)$$

Dividing (8.2.3) by $a_n$ and summing from $n_5$ to $(n - 1)$, and applying condition (H2), we obtain $z_n \to -\infty$ as $n \to \infty$. This contradiction shows that $\Delta^2(a_n \Delta z_n) > 0$ for sufficiently large $n$ which completes the proof. \hfill $\square$

**Lemma 8.2.2.** Assume that (H1)-(H4) hold. Let $y_n$ be an eventually positive solution of equation (8.1.1) and suppose that case (I) of lemma 8.2.1 holds. Then there exists $n_1 \geq n_0$ sufficiently large such that

$$\Delta z_{n-\sigma} \geq \frac{\delta_{n-\sigma} c_n}{a_{n-\sigma}} \Delta^2(a_n \Delta z_n) \text{ for } n \geq n_1 \quad (8.2.4)$$

where $\delta_n = \sum_{s=n_0}^{n-1} \frac{1}{c_s}$.

**Proof.** From case(I) of Lemma 8.2.1 and equation (8.1.1), we have for $n \geq n_1$,

$$a_n \Delta z_n > 0; \ c_n \Delta^2(a_n \Delta z_n) > 0 \text{ and } \Delta(c_n \Delta^2(a_n \Delta z_n)) \leq 0. \text{ Since } \sum_{s=n_1}^{n-1} \Delta^2(a_s \Delta z_s) = \Delta(a_n \Delta z_n) - \Delta(a_{n_1} \Delta z_{n_1}) \text{ for } n \geq n_1.$$  

That is $\Delta(a_n \Delta z_n) = \Delta(a_{n_1} \Delta z_{n_1}) + \sum_{s=n_1}^{n-1} \Delta^2(a_s \Delta z_s) \geq \Delta^2(a_n \Delta z_n). \quad (8.2.5)$
Summing (8.2.5) from \( n_2 \) to \( (n - 1) \), for \( n_2 \geq n_1 \)

\[
a_n \Delta z_n \geq a_{n_2} \Delta z_{n_2} + \sum_{s=n_1}^{n-1} \frac{c_s \Delta^2(a_s \Delta z_s)}{c_s} \geq c_n \delta_n \Delta^2(a_n \Delta z_n).\]

The last inequality and equation (8.2.5), we have

\[
a_n \Delta z_n \geq c_n \delta_n \Delta^2(a_n \Delta z_n). \tag{8.2.6}
\]

Since \( \Delta(c_n \Delta^2(a_n \Delta z_n)) \leq 0 \), we get

\[
a_n \Delta z_n \geq c_n \delta_n \Delta^2(a_n \Delta z_n). \tag{8.3.1}
\]

Furthermore, assume that there exists a positive sequence \( \{\phi_n\}_{n=n_0}^{\infty} \) such that

\[
\lim_{n \to \infty} \sup_n \sum_{s=n_2}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} \left( \sum_{u=t}^{\infty} \frac{1}{c_u} \sum_{i=u}^{\infty} q_i \right) = \infty. \tag{8.3.2}
\]

Then equation (8.1.1) is oscillatory.

**Proof.** Let \( \{y_n\} \) be a non-oscillatory solution of equation (8.1.1), without loss of generality, we may assume that \( y_n > 0 \), \( y_{n-\tau} > 0 \) and \( y_{n-\sigma} > 0 \) for \( n \geq n_1 \), where \( n_1 \geq n_0 \) is chosen so large that Lemma 8.2.1 and Lemma 8.2.2 hold. We shall

8.3 Results Related to Oscillatory Behavior

**Theorem 8.3.1.** Assume that (H1)-(H4) holds and

\[
\lim_{n \to \infty} \sup_n \sum_{s=n_2}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} \left( \sum_{u=t}^{\infty} \frac{1}{c_u} \sum_{i=u}^{\infty} q_i \right) = \infty. \tag{8.3.1}
\]

Furthermore, assume that there exists a positive sequence \( \{\phi_n\}_{n=n_0}^{\infty} \) such that

\[
\lim_{n \to \infty} \sup_n \sum_{s=n_2}^{n-1} \left( K \phi_s q_s - \frac{(\Delta \phi_s)^2 a_{s-\sigma}}{4 \phi_{s+1} \delta_{s-\sigma}} \right) = \infty. \tag{8.3.2}
\]

Then equation (8.1.1) is oscillatory.
consider only this case, because the proof when \( y_n < 0 \) is similar. According to Lemma 8.2.1, there are two possible cases.

Case(I) : \( \Delta z_n > 0 \) for \( n \geq n_1 \geq n_0 \).

In this case , we define the function \( w_n \) by

\[
w_n = \phi_n \frac{c_n \Delta^2(a_n \Delta z_n)}{z_{n-\sigma}}; n \geq n_1. \tag{8.3.3}
\]

Then by equation (8.1.1) and Lemma 8.2.2, we have

\[
\Delta w_n \leq -K\phi_n q_n + \frac{\Delta \phi_n}{\phi_{n+1}} w_{n+1} - \frac{\delta_{n-\sigma}}{\phi_{n+1}a_{n-\sigma}} w_{n+1}^2, \tag{8.3.4}
\]

which implies

\[
\Delta w_n < -K\phi_n q_n + \frac{(\Delta \phi_n)^2 a_{n-\sigma}}{4\phi_{n+1}\delta_{n-\sigma}} - \left( \frac{\delta_{n-\sigma}}{\phi_{n+1}a_{n-\sigma}} w_{n+1} - \frac{\Delta \phi_n}{2\phi_{n+1}} \frac{\phi_{n+1}a_{n-\sigma}}{\delta_{n-\sigma}} \right)^2, \tag{8.3.5}
\]

and hence

\[
\Delta w_n < - \left( K\phi_n q_n - \frac{(\Delta \phi_n)^2 a_{n-\sigma}}{4\phi_{n+1}\delta_{n-\sigma}} \right). \tag{8.3.6}
\]

Summing equation (8.3.6),we have for \( n \geq n_2 \),

\[
w_n < w_{n_2} - \sum_{s=n_2}^{n-1} \left( K\phi_s q_s - \frac{(\Delta \phi_s)^2 a_{s-\sigma}}{4\phi_{s+1}\delta_{s-\sigma}} \right). \tag{8.3.7}
\]

Letting \( n \to \infty \), in view of equation (8.3.2), we have \( w_n \to -\infty \), which is a contradiction.

Case(II): \( \Delta z_n < 0 \), for \( n \geq n_1 \geq n_0 \).

This implies that \( y_n \) is positive and decreasing function. Summing equation (8.1.1) from \( n_1 \) to \( (n - 1) \ (n \geq n_1) \), we obtain,

\[
c_n \Delta^2(a_n \Delta z_n) - c_{n_1} \Delta^2(a_{n_1} \Delta z_{n_1}) + \sum_{s=n_1}^{n-1} q_s y_{s-\sigma} \leq 0.
\]
From Lemma 8.2.1, since $c_n \Delta^2(a_n \Delta z_n) > 0$ and decreasing, we have

$$-c_n \Delta^2(a_n \Delta z_n) + K \sum_{i=n_1}^{\infty} q_i y_{i-\sigma} \leq 0.$$  

This implies that

$$\Delta^2(a_n \Delta z_n) + K \frac{1}{c_n} \sum_{i=n_1}^{n-1} q_i y_{i-\sigma} \leq 0.$$  

Summing again from $n$ to $\infty$, we have

$$-\Delta(a_n \Delta z_n) + K \sum_{u=n}^{\infty} \sum_{i=u}^{\infty} q_i y_{i-\sigma} \leq 0.$$  

Summing from $n$ to $\infty$, using $\Delta z_n < 0$, we have

$$a_n \Delta z_n + K \sum_{t=n_1}^{\infty} \left( \sum_{u=t}^{\infty} \sum_{i=u}^{\infty} q_i y_{i-\sigma} \right) \leq 0.$$  

Summing from $n_1$ to $(n-1)$, we obtain

$$z_n - z_{n_1} + K \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} \left( \sum_{u=t}^{\infty} \sum_{i=u}^{\infty} q_i y_{i-\sigma} \right) \leq 0.$$  

Hence, using the fact that $y_n$ is decreasing, we have

$$y_n + b_n y_{n-\tau} - y_{n_1} - b_{n_1} y_{n_1-\tau} + K y_{n_1} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} \left( \sum_{u=t}^{\infty} \sum_{i=u}^{\infty} q_i \right) \leq 0,$$

which implies

$$\lim_{n \to \infty} \sup_{n=n_0} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} \left( \sum_{u=t}^{\infty} \sum_{i=u}^{\infty} q_i \right) < \infty,$$

which contradicts the equation (8.3.1). Hence the theorem was proved. \(\square\)

**Theorem 8.3.2.** Let all the assumption of Theorem 8.3.1 holds except the condition (8.3.2), which changed to

$$\lim_{m \to \infty} \sup_{n=n_0} \frac{1}{m^r} \sum_{n=n_0}^{m-1} (m-n)^r \left( K \phi_n q_n - \frac{(\Delta \phi_n)^2 a_{n-\sigma}}{4 \phi_{n+1} \delta_{n-\sigma}} \right) = \infty, \quad r \geq 1. \quad (8.3.8)$$

Then equation (8.1.1) is oscillatory.
Proof. Proceeding as in the proof of Theorem 8.3.1, we assume that equation (8.1.1) has non-oscillatory solution, say \( y_n > 0, y_{n-\tau} > 0 \) and \( y_{n-\sigma} > 0 \) for \( n \geq n_1 \).

Where \( n_1 \geq n_0 \) is chosen so large that Lemma 8.2.1 and Lemma 8.2.2 hold. We shall consider only this case, because the proof when \( y_n < 0 \) is similar.

According to Lemma 9.2.1, there are two possible cases.

If the case(I) holds, then by defining again \( w_n \) by equation (8.3.3) as in the Theorem 8.3.1, we have \( w_n > 0 \) and (8.3.6) holds. From (8.3.6) we have for \( n \geq n_1 \)

\[
\sum_{n=n_1}^{m-1} (m-n)^r \left( K\phi_n q_n - \frac{(\Delta \phi_n)^2 a_{n-\sigma}}{4\phi_n+\delta_{n-\sigma}} \right) < - \sum_{n=n_1}^{m-1} (m-n)^r \Delta w_n, \tag{8.3.9}
\]

Since

\[
\sum_{n=n_1}^{m-1} (m-n)^r \Delta w_n = r \sum_{n=n_1}^{m-1} (m-n)^{r-1} w_n - w_{n_1} (n-n_1)^r. \tag{8.3.10}
\]

We get,

\[
\frac{1}{m^r} \sum_{n=n_1}^{m-1} (m-n)^r G_n \leq w_{n_1} \left( \frac{m-n_1}{m} \right)^r - \frac{r}{m^r} \sum_{n=n_1}^{m-1} (m-n)^{r-1} w_n, \tag{8.3.11}
\]

where \( G_n = K\phi_n q_n - \frac{(\Delta \phi_n)^2 a_{n-\sigma}}{4\phi_n+\delta_{n-\sigma}} \). That is,

\[
\frac{1}{m^r} \sum_{n=n_1}^{m-1} (m-n)^r G_n \leq w_{n_1} \left( \frac{m-n_1}{m} \right)^r. \tag{8.3.12}
\]

Then

\[
\lim_{m \to \infty} \sup \frac{1}{m^r} \sum_{n=n_1}^{m-1} (m-n)^r G_n \to w_{n_1}
\]

which contradicts the condition (8.3.8).
If the Case(II) holds, we come back to the proof of the second part of Theorem 8.3.1 and hence the proof is omitted. This completes the proof.

Next, we present some new oscillation results for equation (8.1.1), we introduce a double sequence \( \{H(m, n) : m \geq n \geq 0\} \) such that

(i) \( H(m, m) = 0, \) for \( m \geq 0, \)

(ii) \( H(m, n) > 0, \) for \( m > n \geq 0 \)

and (iii) \( -\Delta_2 H(m, n) = h(m, n)\sqrt{H(m, n)}, m > n \geq 0, \) where\( \Delta_2 H(m, n) = H(m, n + 1) - H(m, n) \geq 0 \) for \( m \geq n \geq 0 \)

\( \square \)

**Theorem 8.3.3.** Assume that (H1)-(H4) holds. Furthermore, assume that there exists sequence \( \{\phi_n\}_{n=n_0}^{\infty} \) and \( H \) such that

\[
\lim_{m \to \infty} \sup_{H(m, n_0)} \frac{1}{m} \sum_{n=n_0}^{m-1} H(m, n) \left( K\phi_n q_n - \frac{\phi_{n+1} a_{n-\sigma} G^2(m, n)}{4\delta_{n-\sigma}} \right) = \infty, \tag{8.3.13}
\]

where \( G(m, n) = \frac{h(m, n)}{\sqrt{H(m, n)}} - \frac{\Delta \phi_n}{\phi_{n+1}}. \tag{8.3.14} \)

Then equation (8.1.1) is oscillatory.

**Proof.** Let \( y_n \) be a non-oscillatory solution of equation (8.1.1). Let us first assume that \( y_n \) is eventually positive and that \( y_n > 0, y_{n-\tau} > 0 \) and \( y_{n-\sigma} > 0 \) for \( n \geq n_1. \)

The case where \( y_n \) is eventually negative with similarly and is omitted. As in the proof of Lemma 8.2.1 there are two possible cases.

Let the Case(I) hold: Again, defining \( w_n \) as in (8.3.3), we obtain (8.3.4).

Let us denote \( \gamma_n = \frac{\Delta \phi_n}{\phi_{n+1}} \) and \( R_n = \frac{\delta_{n-\sigma}}{\phi_{n+1} a_{n-\sigma}}. \)

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Then from (8.3.4), we get
\[
\sum_{n=n_1}^{m-1} H(m, n)K\phi_nq_n \leq \sum_{n=n_1}^{m-1} H(m-n) \left[-\Delta w_n + \gamma_n w_{n+1} - R_n w_{n+1}^2\right]
\]
\[
= \left[-H(m, n)w_n\right]_{n=n_1}^{m-1} + \sum_{n=n_1}^{m-1} \left\{ \Delta_2 H(m, n)w_{n+1} + H(m, n) \left[\gamma_n w_{n+1} - R_n w_{n+1}^2\right]\right\}
\]
\[
= -\sum_{n=n_1}^{m-1} \left[\sqrt{H(m,n)} \left(h(m,n) - \sqrt{H(m,n)} \gamma_n\right) w_{n+1} + H(m, n)R_n w_{n+1}^2\right]
\]
\[
+ H(n, n_1)w_{n_1}
\]
\[
= -\sum_{n=n_1}^{m-1} H(m, n) \left[\sqrt{R_n} w_{n+1} + \frac{1}{2} \frac{G(m, n)}{\sqrt{R_n}}\right]^2 + \sum_{n=n_1}^{m-1} \frac{G^2(m, n)H(m, n)}{4R_n}
\]
\[
+ H(m, n_1)w_{n_1}.
\] (8.3.15)

It follows that
\[
\frac{1}{H(m, n_1)} \sum_{n=n_1}^{m-1} H(m, n) \left(K\phi_nq_n - \frac{G^2(m, n)}{4R_n}\right) \leq w_{n_1}.
\] (8.3.16)

This contradicts (8.3.13). If the case(I) holds, we come back to the proof of the second part of Theorem 8.3.1 and hence it is omitted. The proof is complete. \(\Box\)

8.4 Examples

Example 8.4.1. Consider the difference equation
\[
\Delta \left( n\Delta^2 \left( n\Delta \left( y_n + (n + 2)y_{n-2}\right)\right)\right) + \frac{16n^4 + 112n^3 + 276n^2 + 306n + 71}{(n - 3)(5n^2 - 30n + 54)} \left(9y_{n-3} + 5y_{n-3}^3\right) = 0, \quad n \geq 4. \quad (E.8.1)
\]

Here \(a_n = n, b_n = n + 2, c_n = n, q_n = \frac{16n^4 + 112n^3 + 276n^2 + 306n + 71}{(n - 3)(5n^2 - 30n + 54)},\)

\(\phi_n = \frac{1}{n^2}, \quad \sum_{n=4}^{\infty} \frac{1}{a_n} = \sum_{n=4}^{\infty} \frac{1}{n} = \infty, \quad \sum_{n=4}^{\infty} \frac{1}{c_n} = \sum_{n=4}^{\infty} \frac{1}{n} = \infty,\)

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\[
\sum_{n=4}^{\infty} q_n = \sum_{n=4}^{\infty} \frac{16n^4 + 112n^3 + 276n^2 + 306n + 71}{(n - 3)(5n^2 - 30n + 54)} = \infty,
\]
\[
\sum_{n=4}^{\infty} \frac{1}{a_n} \sum_{n=4}^{\infty} \left( \sum_{n=4}^{\infty} \frac{1}{c_n} \sum_{n=4}^{\infty} q_n \right) = \infty,
\]
\[
\sum_{n=4}^{\infty} \left( K\phi_n q_n - \frac{(\Delta \phi_n)^2 q_{n-\sigma}}{4\phi_{n+1}\delta_{n-\sigma}} \right)
\]
\[
= \sum_{n=4}^{\infty} \left( K \frac{16n^4 + 112n^3 + 276n^2 + 306n71}{n^2(n - 3)(5n^2 - 30n + 54)} - \frac{(2n + 1)^2(n - 3)}{4n^2(n + 1)^2 \sum_{s=4}^{n-1} \frac{1}{s - 3}} \right) = \infty,
\]

All conditions of Theorem 8.3.1 are satisfied for \(H(m,n) = m - n\), \(h(m,n) = \frac{1}{\sqrt{m - n}}\). Hence equation (E.8.1) is oscillatory. In fact \(y_n = n(-1)^n\) is a solution of equation (E.8.1).

Example 8.4.2. Consider the difference equation

\[
\Delta \left( n\Delta^2 \left( n\Delta \left( y_n + (n + 2)y_{n-2} \right) \right) \right) + \frac{16n^4 + 112n^3 + 276n^2 + 306n71}{(n - 3)(5n^2 - 30n + 54)} \left( 9y_{n-3} + 5y_{n-3}^2 \right) = 0, \ n \geq 4. \quad (E.8.2)
\]

Here \(a_n = n\), \(b_n = n + 2\), \(c_n = n\), \(q_n = \frac{16n^4 + 112n^3 + 276n^2 + 306n + 71}{(n - 3)(5n^2 - 30n + 54)}\), \(\phi_n = 5n^2 - 30n + 54\), \(\sum_{n=4}^{\infty} \frac{1}{a_n} = \sum_{n=4}^{\infty} \frac{1}{n} = \infty\), \(\sum_{n=4}^{\infty} \frac{1}{c_n} = \sum_{n=4}^{\infty} \frac{1}{n} = \infty\), \(\frac{h(m,n)}{\sqrt{H(m,n)}} - \frac{\Delta \phi_n}{\phi_{n+1}} = \frac{1}{(m - n)^2} - \frac{10n - 25}{5n^2 - 20n + 29}\).

\[
G(m,n) = \frac{h(m,n)}{\sqrt{H(m,n)}} - \frac{\Delta \phi_n}{\phi_{n+1}} = \frac{1}{(m - n)^2} - \frac{10n - 25}{5n^2 - 20n + 29}
\]

\[
\lim_{m \to \infty} \sup H(m,n_0) \sum_{n=4}^{m-1} H(m,n) \left( K\phi_n q_n - \frac{\phi_{n+1} a_{n-\sigma} G^2(m,n)}{4\delta_{n-\sigma}} \right)
\]

\[
= \lim_{m \to \infty} \sup (m - 4)^2 \sum_{n=4}^{m-1} (m - n)^2 \left( K \frac{16n^4 + 112n^3 + 276n^2 + 306n + 71}{(n - 3)} - \frac{(5n^2 - 20n + 29)(n - 3)G^2(m,n)}{4} \right) = \infty.
\]
and

$$\lim_{m \to \infty} \sup_{m^r} \sum_{n=4}^{n=m-1} (m-n)^r \left( K \phi_n q_n - \frac{(\Delta \phi_n)^2 a_{n-\sigma}}{4 \phi_{n+1} \delta_{n-\sigma}} \right)$$

$$= \lim_{m \to \infty} \sup_{m^2} \sum_{n=4}^{n=m-1} (m-n)^2 \left( K \frac{16n^4 + 112n^3 + 276n^2 + 306n + 71}{(n-3)} - \frac{(n-3)(10n - 25)^2}{4(5n^2 - 20n + 29) \sum_{s=4}^{n-1} \frac{1}{(n-3)}} \right) = \infty.$$ 

All conditions of Theorem 8.3.2 and 8.3.3 are satisfied for $H(m,n) = (m-n)^2$, $h(m,n) = \frac{1}{m-n}$. Hence equation (E.8.2) is oscillatory. In fact $y_n = n(-1)^n$ is a solution of equation (E.8.2).