Chapter 6

OSCILLATORY BEHAVIOR OF THIRD ORDER NONLINEAR DIFFERENCE EQUATIONS

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6.1 Introduction

In this chapter, we are concerned with the oscillatory behavior of all the solutions of a third order nonlinear difference equation of the form

$$\Delta \left( a_n \left( \Delta^2 x_n \right)^\alpha \right) + q_n f \left( x_{\sigma(n)} \right) = 0, \ n \in N, \quad (6.1.1)$$

where the following conditions are assumed to be hold.

(H1) \{a_n\} and \{q_n\} are real positive sequences, \(\sigma(n) \leq n, \ \lim_{n \to \infty} \sigma(n) = \infty\).

(H2) \(\sum_{s=n_0}^\infty a_s^{-\frac{1}{\alpha}} < \infty\).
(H3) $\alpha$ is a quotient of odd positive integers.

(H4) $f : \mathbb{R} \to \mathbb{R}$ is continuous, $xf(x) > 0$ for $x \neq 0$ and $-f(-xy) \geq f(xy) \geq f(x)f(y)$ for $xy > 0$.

By a solution of equation (6.1.1), we mean a real sequence \( \{x_n\} \) satisfying (6.1.1) for \( n \in \mathbb{N} \).

### 6.2 Basic Lemmas

**Lemma 6.2.1.** Assume that (H2) holds. Let \( x_n \) be a positive solution of equation (6.1.1). Then either

(i) \( \Delta^2 x_n > 0 \), eventually and \( x_n \) is either strongly increasing or strongly decreasing, or

(ii) \( \Delta^2 x_n < 0 \), eventually and \( x_n \) is strongly increasing.

**Proof.** Let \( x_n \) be a nonoscillatory solution of equation (6.1.1). We may assume that \( x_n > 0 \), eventually (if it is an eventually negative, the proof is similar). Then \( \Delta \left( a_n (\Delta^2 x_n)\alpha \right) < 0 \), eventually. Thus, \( a_n (\Delta^2 x_n)\alpha \) is decreasing and of one sign and it follows from hypothesis (H1) and (H2) that there exists a \( n_1 \geq n_0 \) such that \( \Delta^2 x_n \) is of fixed sign for \( n \geq n_1 \). If we have \( \Delta^2 x_n > 0 \), then \( \Delta x_n \) is increasing and then either \( x_n \Delta x_n < 0 \) or \( x_n \Delta x_n > 0 \) holds, eventually.

On the other hand, if \( \Delta^2 x_n < 0 \) then \( \Delta x_n \) is decreasing, hence \( \Delta x_n \) is of fixed sign. If we have \( \Delta x_n < 0 \), then \( \lim_{n \to \infty} x_n = -\infty \). This contradicts the positivity of \( x_n \). Whereupon \( \Delta x_n > 0 \). The proof is complete.

The following criterion eliminates case (ii) of Lemma 6.2.1.
Lemma 6.2.2. Let $x_n$ be a positive solution of equation (6.1.1). If

$$\sum_{t=n_0}^{\infty} \left[ \frac{1}{a_t} \sum_{s=n_0}^{t-1} q_s f(\sigma(s)) f \left( \sum_{v=\sigma(s)}^{\infty} a_v^{-\frac{1}{\alpha}} \right) \right] = \infty,$$  \hspace{2cm} (6.2.1)

then $x_n$ does not satisfy case (ii) of Lemma 6.2.1.

Proof. Let $x_n$ be a positive solution of equation (6.1.1). We assume that $x_n$ satisfies case (ii) of Lemma 6.2.1. That is $\Delta^2 x_n < 0$ and $\Delta x_n > 0$, eventually. Then there exists a $n_1 \geq n_0$ and a constant $k$, $0 < k < 1$ such that $x_n \geq kn \Delta x_n$ for $n \geq n_1$. Consequently,

$$x_{\sigma(n)} \geq k \sigma(n) \Delta x_{\sigma(n)}, \text{ for } n \geq n_2 \geq n_1. \hspace{2cm} (6.2.2)$$

Now equation (6.1.1), in view of (H4) and (6.2.2), implies

$$\Delta \left( a_n (\Delta^2 x_n)^{\alpha} \right) + f(k)q_n f(\sigma(n)) f \left( \Delta x_{\sigma(n)} \right) \leq 0.$$  

Summation of this inequality yields

$$f(k) \sum_{s=n_2}^{n_1} q_s f(\sigma(s)) f \left( \Delta x_{\sigma(s)} \right) \leq a_{n_2} (\Delta^2 x_{n_2})^\alpha - a_n (\Delta^2 x_n)^\alpha. \hspace{2cm} (6.2.3)$$

On the other hand, Since $-a_n^{\frac{1}{\alpha}} (\Delta^2 x_n)$ is increasing, there exists a constant $m > 0$ such that

$$-a_n^{\frac{1}{\alpha}} \Delta^2 x_n \geq m, \text{ for } n \geq n_2, \hspace{2cm} (6.2.4)$$

which implies

$$\Delta x_{\sigma(n)} \geq \sum_{s=\sigma(n)}^{\infty} -a_s^{\frac{1}{\alpha}} \Delta^2 x_s a_s^{-\frac{1}{\alpha}} \geq m \sum_{s=\sigma(n)}^{\infty} a_s^{-\frac{1}{\alpha}}. \hspace{2cm} (6.2.5)$$

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Combining (6.2.5) together with (6.2.3), and taking into account (H3), we get
\[
c \left[ \frac{1}{a_n} \sum_{s=n_2}^{n-1} q_s f(\sigma(s)) f \left( \sum_{v=\sigma(s)}^{\infty} a_v^{\frac{1}{\alpha}} \right) \right]^{\frac{1}{\alpha}} = -\Delta^2 x_n, \quad (6.2.6)
\]
where \( c = (f(m)f(k))^{\frac{1}{\alpha}} \). Summing (6.2.6) from \( n_3 \) to \( n - 1 \), we have
\[
c \sum_{t=n_3}^{n-1} \left[ \frac{1}{a_t} \sum_{s=n_2}^{t-1} q_s f(\sigma(s)) f \left( \sum_{v=\sigma(s)}^{\infty} a_v^{\frac{1}{\alpha}} \right) \right]^{\frac{1}{\alpha}} \leq \Delta x_{n_3}.
\]
Letting \( n \to \infty \) we get a contradiction to condition (6.2.1). Therefore, we have eliminated case (ii) of Lemma 6.2.1.

Lemma 6.2.3. Assume that \( x_n \) is strongly decreasing solution of equation (6.1.1).
If
\[
\sum_{u=n_0}^{\infty} \sum_{v=n}^{\infty} a_u^{\frac{1}{\alpha}} \left( \sum_{s=u}^{\infty} q_s \right)^{\frac{1}{\alpha}} = \infty, \quad (6.2.7)
\]
then \( x_n \) tends to zero as \( n \to \infty \).

Proof. We may assume that \( x_n \) is positive. It is clear that there exists a finite \( \lim_{n \to \infty} x_n = L \). We shall prove that \( L = 0 \). Assume that \( L > 0 \). Summing equation (6.1.1) from \( n \) to \( \infty \) and using \( x_{\sigma(n)} > L \) and (H3), we obtain
\[
a_n \left( \Delta^2 x_n \right)^{\alpha} \geq \sum_{s=n}^{\infty} q_s f \left( x_{\sigma(n)} \right) \geq f(L) \sum_{s=n}^{\infty} q_s,
\]
which implies
\[
\Delta^2 x_n \geq a_n^{\frac{1}{\alpha}} L_1 \left( \sum_{s=n}^{\infty} q_s \right)^{\frac{1}{\alpha}},
\]
where \( L_1 = f^{\frac{1}{\alpha}}(L) > 0 \). Summing the last inequality from \( n \) to \( \infty \), we get
\[
-\Delta x_n \geq L_1 \sum_{u=n}^{\infty} a_u^{\frac{1}{\alpha}} \left( \sum_{s=u}^{\infty} q_s \right)^{\frac{1}{\alpha}}.
\]

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Now summing from \( n_1 \) to \( n - 1 \), we arrive at

\[
x_{n_1} \geq L_1 \sum_{v=n_1}^{n-1} \sum_{u=v}^{\infty} a_u^{-\frac{1}{\alpha}} \left( \sum_{s=u}^{\infty} q_s \right)^{\frac{1}{\alpha}}.
\]

Letting \( n \to \infty \) we have a contradiction with (6.2.7) and so we have proved that

\[
\lim_{n \to \infty} x_n = 0.
\]

\[\square\]

### 6.3 Results Related to Oscillatory Behavior-I

**Theorem 6.3.1.** Let (6.2.1) hold. If

\[
\Delta y_n + q_n f \left[ \sum_{u=n_0}^{\sigma(n) - 1} (\sigma(n) - u) a_u^{-\frac{1}{2}} \right] f \left[ y_{\sigma(n)}^{\frac{1}{2}} \right] = 0,
\]

is oscillatory, then every solution of equation (6.1.1) is either oscillatory or strongly decreasing.

**Proof.** Let \( x_n \) be a nonoscillatory solution of equation (6.1.1). We may assume that \( x_n > 0 \) for \( n \geq n_0 \). From Lemma 6.2.2, we see that \( \Delta x_n > 0 \) and \( x_n \) is either strongly increasing or strongly decreasing.

Assume that \( x_n \) is strongly increasing, that is \( \Delta x_n > 0 \), eventually. Using the fact that \( a_n (\Delta^2 x_n)^{\alpha} \) is decreasing, we are lead to

\[
\Delta x_n \geq \sum_{s=n_1}^{n-1} \Delta^2 x_s = \sum_{s=n_1}^{n-1} a_s^{-\frac{1}{\alpha}} \left[ a_s (\Delta^2 x_s)^{\alpha} \right]^{\frac{1}{\alpha}}
\]

\[
\geq \left[ a_n (\Delta^2 x_n)^{\alpha} \right]^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} a_s^{-\frac{1}{\alpha}}.
\]

(6.3.2)

Summing (6.3.2) from \( n_1 \) to \( n - 1 \), we have

\[
x_n \geq \sum_{s=n_1}^{n-1} \left[ a_s (\Delta^2 x_s)^{\alpha} \right]^{\frac{1}{\alpha}} \sum_{u=n_1}^{s-1} a_u^{-\frac{1}{\alpha}}
\]

\[
\geq \left[ a_n (\Delta^2 x_n)^{\alpha} \right]^{\frac{1}{\alpha}} \sum_{u=n_1}^{s-1} (n - u) a_u^{-\frac{1}{\alpha}}.
\]
There exists an $n_2 \geq n_1$ such that for all $n \geq n_2$, one gets

$$x_{\sigma(n)} \geq y_{\sigma(n)} \sum_{u=n_2}^{\sigma(n)-1} [\sigma(n) - u] a_u^{-\frac{1}{\alpha}}, \quad (6.3.3)$$

where $y_n = a_n (\Delta^2 x_n)^\alpha$. Combining (6.3.3) together with (6.1.1), we see that

$$-\Delta y_n = q_n f(x_{\sigma(n)}) \geq q_n f \left[ \frac{1}{\alpha} \sum_{u=n_2}^{\sigma(n)-1} [\sigma(n) - u] a_u^{-\frac{1}{\alpha}} \right]$$

$$\geq q_n f \left[ \sum_{u=n_2}^{\sigma(n)-1} [\sigma(n) - u] a_u^{-\frac{1}{\alpha}} \right] f \left[ y_{\sigma(n)}^{\frac{1}{\alpha}} \right],$$

where we have used (H3). Thus, $y_n$ is positive and decreasing solution of the difference inequality

$$\Delta y_n + q_n f \left[ \sum_{u=n_2}^{\sigma(n)-1} (\sigma(n) - u) a_u^{-\frac{1}{\alpha}} \right] f \left[ y_{\sigma(n)}^{\frac{1}{\alpha}} \right] \leq 0.$$

Hence we conclude that the corresponding difference equation (6.3.1) is also has a positive solution, which is a contradiction to the oscillation of (6.3.1). Therefore $x_n$ is strongly decreasing.

Adding an additional condition, we achieve stronger asymptotic behavior of nonoscillatory solution of (6.1.1).

\[\square\]

**Theorem 6.3.2.** Assume that (6.2.1) and (6.2.7) holds. If the equation (6.3.1) is oscillatory then every solution of equation (6.1.1) is oscillatory or tends to zero as $n \to \infty$.

**Proof.** Combining Theorem 6.3.1 and Lemma 6.2.3, we get the proof of this theorem. \[\square\]
Theorem 6.3.3. Assume that (6.2.7) hold and

\[ \sum_{t=n_0}^{\infty} \left[ \frac{1}{a_t} \sum_{s=n_0}^{t-1} q_s \sigma^\beta(s) \left( \sum_{v=\sigma(s)}^{\infty} a_v^{-\frac{1}{\alpha}} \right)^\beta \right]^{\frac{1}{\alpha}} = \infty. \quad (6.3.4) \]

Assume that \( \beta \) is a quotient of odd positive integer. If

\[ \Delta y_n + q_n \left[ \sigma(n) - 1 \right] \sum_{s=n_0}^{\sigma(n)-1} (\sigma(n) - s) a_s^{-\frac{1}{\alpha}} y_{\sigma(n)}^\beta = 0, \]

is oscillatory then the solution \( x_n \) of the equation

\[ \Delta \left( a_n \left( \Delta^2 x_n \right)^\alpha \right) + q_n x_{\sigma(n)}^\beta = 0, \]

is oscillatory or the solution tends to zero as \( n \to \infty \).

Proof. Proof of this theorem is similar to the proofs of Theorem 6.3.1 and Theorem 6.3.2, hence the details are omitted. \qed

6.4 Special Case of Equation (6.1.1)

In this section, we are concerned with the oscillatory behavior of all the solutions of the third order difference equation of the form

\[ \Delta \left( \frac{1}{a_n} \Delta^2 y_n \right) + p_n f(y_{\sigma(n)}) = 0, \quad n \in \mathbb{N}, \quad (6.4.1) \]

which is a special case of equation (6.1.1), where the following conditions are assumed to be hold.

(h1) \( \{a_n\}, \{p_n\} \) and \( \{\sigma(n)\} \) are real positive sequences and \( p_n \neq 0 \) for infinitely many values of \( n \).

(h2) \( \sigma(n) \leq n \) and \( \lim_{n \to \infty} \sigma(n) = \infty \).
(h3) \[ \sum_{s=n_0}^{n-1} R_n = \sum_{s=n_0}^{n-1} a_s \to \infty \text{ as } n \to \infty. \]

(h4) \( f : \mathbb{R} \to \mathbb{R} \) is continuous such that \( xf(x) > 0 \) for \( x \neq 0 \) and \( \frac{f(x)}{x} \geq L > 0. \)

### 6.4.1 Results Related to Oscillatory Behavior-II

In this section, we present some sufficient conditions for oscillation of equation (6.4.1).

**Theorem 6.4.1.** Assume that (h3) holds and

\[
\Delta \sigma(n) \geq 0,
\]

\[
\sum_{s=n_0}^{\infty} \left( LR_{\sigma(s)} p_s - \frac{(\Delta R_{\sigma(s)})^2}{4R_{\sigma(s)} a_{s+1}(s-n_1)} \right) = \infty. \tag{6.4.2}
\]

Then equation (6.4.1) is oscillatory.

**Proof.** Let \( \{y_n\} \) be a non-oscillatory solution of equation (6.4.1). Without loss of generality, we may assume that \( y_n > 0, y_{\sigma(n)} > 0 \) for \( n \geq n_1. \)

From (6.4.1) we have

\[
\Delta \left( \frac{1}{a_n} \Delta^2 y_n \right) < 0, \text{ for } n \geq n_1.
\]

since \( \Delta \left( \frac{1}{a_n} \Delta^2 y_n \right) \) is non-increasing there exists a non negative constant \( k \) and \( n_2 \geq n_1 \) such that

\[
\Delta \left( \frac{1}{a_n} \Delta^2 y_n \right) \leq -k, \text{ for } n \geq n_2.
\]
Summing the inequality from \( n_2 \) to \( (n-1) \), we obtain
\[
\frac{1}{a_n} \Delta^2 y_n \leq \frac{1}{a_{n_2}} \Delta^2 y_{n_2} - k(n - n_2).
\]

Letting \( n \to \infty \), we have \( \frac{1}{a_n} \Delta^2 y_n \to -\infty \). Thus, there is an integer \( n_3 \geq n_2 \) such that \( n \geq n_3 \),
\[
\frac{1}{a_n} \Delta^2 y_n \leq \frac{1}{a_{n_3}} \Delta^2 y_{n_3} < 0
\]
That is
\[
\Delta^2 y_n \leq -k_1 a_n, \; k_1 > 0.
\]

Summing the last inequality from \( n_3 \) to \( (n-1) \), we have
\[
\Delta y_n \leq \Delta y_{n_3} - k_1 \sum_{s=n_3}^{n-1} a_s.
\]

Letting \( n \to \infty \), we have \( \Delta y_n \to -\infty \). Thus, there is an integer \( n_4 \geq n_3 \) such that \( n \geq n_4 \), \( \Delta y_n \leq -k_2 \). Summing the last inequality from \( n_4 \) to \( (n-1) \), we have
\[
y_n \leq y_{n_4} - k_4(n - n_4),
\]
This implies that \( y_n \to -\infty \) as \( n \to \infty \), which is a contradiction to the fact that \( y_n \) is positive. Then \( \Delta \left( \frac{1}{a_n} \Delta^2 y_n \right) > 0 \) and \( \frac{1}{a_n} \Delta^2 y_n > 0 \).

Define \( w_n = \frac{R_{\sigma(n)} \Delta^2 y_n}{f(y_{\sigma(n)}) a_n} \). Then
\[
\Delta w_n = \frac{R_{\sigma(n)} \Delta^2 y_n}{f(y_{\sigma(n)})} \Delta \left( \frac{1}{a_n} \Delta^2 y_n \right) + \frac{\Delta^2 y_{n+1}}{a_{n+1}} \Delta \left( \frac{R_{\sigma(n)}}{f(y_{\sigma(n)})} \right),
\]
which implies
\[
\Delta w_n = \frac{R_{\sigma(n)} \Delta^2 y_n}{f(y_{\sigma(n)})} + \frac{\Delta R_{\sigma(n)}}{R_{\sigma(n+1)}} w_{n+1} - \frac{R_{\sigma(n)} \Delta^2 y_{n+1} \Delta f(y_{\sigma(n)})}{a_{n+1} f(y_{\sigma(n)}) f(y_{\sigma(n+1)})}.
\]

(6.4.3)
Note that,
\[ \Delta y_n = \Delta y_1 + \sum_{s=n_1}^{n-1} \Delta^2 y_n \geq (n - 1 - n_1) \Delta^2 y_n, \quad n \geq n_1 + 1. \]

This implies that
\[ \Delta y_{n+1} \geq (n - n_1) \Delta^2 y_{n+1}; \quad n \geq n_2 = n_1 + 1. \quad (6.4.4) \]

In view of (h2), (h4), equation (6.4.1) and equation (6.4.4), equation (6.4.3) becomes
\[ \Delta w_n \leq -LR_{\sigma(n)} p_n + \frac{\Delta R_{\sigma(n)} w_{n+1}}{R_{\sigma(n+1)}} - (n - n_1) \frac{R_{\sigma(n)} (\Delta^2 y_{n+1})^2}{a_{n+1} \left( f(y_{n+1}) \right)^2}, \]

which implies
\[ \Delta w_n \leq -LR_{\sigma(n)} p_n + \frac{\Delta R_{\sigma(n)} w_{n+1}}{R_{\sigma(n+1)}} - (n - n_1) \frac{R_{\sigma(n)} a_{n+1}}{R_{\sigma(n+1)}^2} w_{n+1}^2. \]

That is,
\[ \Delta w_n \leq -LR_{\sigma(n)} p_n - \left[ \frac{\sqrt{R_{\sigma(n)} a_{n+1} (n - n_1)}}{R_{\sigma(n+1)}} w_{n+1} - \frac{\Delta R_{\sigma(n)}}{2 \sqrt{R_{\sigma(n)} a_{n+1} (n - n_1)}} \right]^2 \]
\[ + \frac{(\Delta R_{\sigma(n)})^2}{4R_{\sigma(n)} a_{n+1} (n - n_1)}. \]

This implies that
\[ \Delta w_n < -\left( LR_{\sigma(n)} p_n - \frac{(\Delta R_{\sigma(n)})^2}{4R_{\sigma(n)} a_{n+1} (n - n_1)} \right). \]

Summing the last inequality from \( n_1 \) to \( (n - 1) \), we have
\[ w_n \leq w_{n_1} - \sum_{s=n_1}^{n-1} \left( LR_{\sigma(s)} p_s - \frac{(\Delta R_{\sigma(s)})^2}{4R_{\sigma(s)} a_{s+1} (s - n_1)} \right). \]

Letting \( n \to \infty \), we have, in view of (6.4.2) that \( w_n \to -\infty \) as \( n \to \infty \), which contradicts \( w_n > 0 \) and the proof is complete.
6.5 Examples

Example 6.5.1. Consider the difference equation

\[ \Delta \left( n^2 \left( \Delta^2 x_n \right) \right) + 4(2n^2 + 2n + 1)x_n^3 = 0. \]  

(E.6.1)

Here \( a_n = n^2 \), \( q_n = 4(2n^2 + 2n + 1) \), \( f(x) = x^3 \), \( \alpha = 1 \), \( \sigma(n) = n \),
\[
\sum_{n=1}^{\infty} a_n^{-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} 4(2n^2 + 2n + 1) = \infty,
\]
\[
\sum_{n=1}^{\infty} \left( \frac{1}{a_n} \sum_{n=1}^{\infty} f(\sigma(n)) f \left( \sum_{n=1}^{\infty} a_n^{-\frac{1}{2}} \right) \right)^{\frac{1}{3}} = \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \sum_{n=1}^{\infty} 4(2n^2 + 2n + 1) n^3 \left( \sum_{n=1}^{\infty} \frac{1}{n^3} \right)^{3} \right)^{\frac{1}{3}} = \infty
\]

and
\[
\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_n^{-\frac{1}{2}} \left( \sum_{n=1}^{\infty} q_n \right)^{\frac{1}{3}} = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^3} \left( \sum_{n=1}^{\infty} 4(2n^2 + 2n + 1) \right)^{\frac{1}{3}} = \infty.
\]

All conditions of Theorem 6.3.1 and Theorem 6.3.2 are satisfied. Hence equation (E.6.1) is oscillatory. One such solution of equation (E.6.1) is \( x_n = (-1)^n \).

Example 6.5.2. Consider the difference equation

\[ \Delta \left( n \left( \Delta^2 x_n \right)^{\frac{1}{2}} \right) + 4^{\frac{1}{3}}(2n + 1)x_n^3 = 0. \]  

(E.6.2)

Here \( a_n = n \), \( q_n = 4^{\frac{1}{3}}(2n + 1) \), \( f(x) = x^3 \), \( \alpha = \frac{1}{3} \), \( \sigma(n) = n^{\frac{1}{3}} \),
\[
\sum_{n=1}^{\infty} a_n^{-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty, \sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} 4^{\frac{1}{3}}(2n + 1) = \infty,
\]
\[
\sum_{n=1}^{\infty} \left( \frac{1}{a_n} \sum_{n=1}^{\infty} f(\sigma(n)) f \left( \sum_{n=1}^{\infty} a_n^{-\frac{1}{2}} \right) \right)^{\frac{1}{3}} = \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{n=1}^{\infty} 4^{\frac{1}{3}}(2n + 1) n \left( \sum_{n=1}^{\infty} \frac{1}{n^3} \right)^{3} \right)^{\frac{1}{3}} = \infty
\]
\[
\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_n^{-\frac{1}{\alpha}} \left( \sum_{n=1}^{\infty} q_n \right)^{\frac{1}{\alpha}} = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^3} \left( \sum_{n=1}^{\infty} 4^{\frac{1}{\alpha}}(2n+1)^3 \right) = \infty.
\]

All conditions of Theorem 6.3.1 and Theorem 6.3.2 are satisfied. Hence every solution of equation (E.6.2) is either oscillatory or strongly decreasing. In fact \( x_n = (-1)^n \) is a solution to equation (E.6.2).

**Example 6.5.3.** Consider the difference equation

\[
\Delta \left( \left( \frac{n(n+1)(n+2)}{2n^3} \right)^{\frac{1}{3}} \left( \Delta^2 x_n \right)^{\frac{1}{3}} \right) + \frac{n^2}{n+1} x_n^3 = 0.
\]  

(E.6.3)

Here \( a_n = \left( \frac{n(n+1)(n+2)}{2n^3} \right)^{\frac{1}{3}} \), \( q_n = \frac{n^2}{n+1} \), \( f(x) = x^3 \), \( \alpha = \frac{1}{3} \), \( \sigma(n) = n \), \( \sum_{n=1}^{\infty} a_n^{-\frac{1}{\alpha}} = \sum_{n=1}^{\infty} \frac{2n^3}{n(n+1)(n+2)} < \infty \), \( \sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} \frac{n^2}{n+1} = \infty \),

\[
\sum_{n=1}^{\infty} \left( \frac{1}{a_n} \sum_{n=1}^{\infty} q_n f(\sigma(n)) f \left( \sum_{n=1}^{\infty} a_n^{-\frac{1}{\alpha}} \right) \right)^{\frac{1}{\alpha}^3} = \sum_{n=1}^{\infty} \left( \frac{2n^3}{n(n+1)(n+2)} \right)^{\frac{1}{3}} \sum_{n=1}^{\infty} \frac{n^2}{n+1} \left( \sum_{n=1}^{\infty} \frac{2n^3}{n(n+1)(n+2)} \right)^{3} = \infty
\]

and

\[
\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_n^{-\frac{1}{\alpha}} \left( \sum_{n=1}^{\infty} q_n \right)^{\frac{1}{\alpha}} = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{2n^3}{n(n+1)(n+2)} \left( \sum_{n=1}^{\infty} \frac{n^2}{n+1} \right)^{3} = \infty.
\]

All conditions of Theorem 6.3.1 and Theorem 6.3.2 are satisfied. Hence every solution of equation (E.6.3) is either oscillatory or tends to zero. In fact \( x_n = \frac{1}{n} \) is a solution to equation (E.6.3).

**Example 6.5.4.** Consider the difference equation

\[
\Delta \left( (n+2)\Delta^2 x_n \right) + \frac{9(2^{2n})(3n+7)}{8(n+2)} x_n^3 = 0.
\]  

(E.6.4)
Here \( a_n = n + 2 \), \( R_n = \frac{1}{a_n} = \frac{1}{n + 2} \), \( p_n = \frac{9(2^{2n})(3n + 7)}{8(n + 2)} \), \( \sigma(n) = n \),
\[
\sum_{n=1}^{\infty} R_n = \sum_{n=1}^{\infty} \frac{1}{n + 2} = \infty \quad \text{and}
\]
\[
\sum_{n=n_0}^{\infty} \left( L R_{\sigma(n)} p_n - \frac{\left( \Delta R_{\sigma(n)} \right)^2}{4 R_{\sigma(n)} a_{n+1}(n - n_1)} \right) = \sum_{n=2}^{\infty} \left( L \frac{9(2^{2n})(3n + 7)}{8(n + 2)} - \frac{1}{4(n + 2)(n + 3)^3(n - 1)} \right) = \infty.
\]

All conditions of theorem 6.4.1 are satisfied. Hence equation (E.6.4) is oscillatory. Infact \( x_n = \frac{(-1)^n}{2^n} \) is a solution to equation (E.6.4).