Chapter 5

OSCILLATION OF THIRD ORDER DIFFERENCE EQUATIONS WITH DELAY

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5.1 Introduction

In this chapter, we are concerned with the oscillation behavior of solutions of third order functional difference equation of the form

\[ \Delta (b(n)\Delta (a(n)\Delta x(n))) + \sum_{i=1}^{m} q_i(n)f(x(\sigma_i(n - \tau))) = h(n), \quad n \in N, \quad (5.1.1) \]

where the following conditions are assumed to be hold.

(H1) \{a(n)\}, \{b(n)\} and \{h(n)\} are real positive sequences.

(H2) \( \Delta b(n) \geq 0 \) and \( \sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \infty, \) for \( n \geq n_0. \)

(H3) \( q_i(n) \neq 0, \) is not identically zero for sufficiently larger \( n, \ i = 0, 1, 2, ..., m. \)
(H4) \( f : \mathbb{R} \to \mathbb{R} \) is continuous and \( xf(x) > 0 \) for all \( x \neq 0 \) and \( K_1 \leq \frac{f(x)}{x} \leq K_2 \).

(H5) \( \rho : \mathbb{N}_0 \to \mathbb{R} \) is continuous for all \( x \neq 0 \in \mathbb{N}_0 \).

(H6) \( \sigma_i(n - \tau) \to \infty \) as \( n \to \infty \), \( i = 0, 1, 2, ..., m \). \( \tau \) is a nonnegative integer.

(H7) \( \sigma_i(n - \tau) < n, \Delta(\sigma_i(n - \tau)) > 0 \), \( i = 0, 1, 2, ..., m \).

5.2 Results Related to Oscillation Behavior

Theorem 5.2.1. Let \( f(x) = x \) and \( b(n) = 0 \). Suppose that there exists a sequence \( \rho(n) > 0 \) such that

\[
\sum_{s=n_2}^{\infty} \left( \rho(s)q(s) - \frac{b(s)\rho(s+1)}{4(s-n_1)\sum_{i=1}^{m} \frac{(\sigma_i(s-\tau)-N)}{a(\sigma_i(s-\tau))}} \right) = \infty,
\]

(5.2.1)

where \( q(n) = \min \{q_1(n), q_2(n), ..., q_m(n)\} \) for every \( N \geq 0 \), and that

\[
\sum_{s=\sigma(n-\tau)}^{n-1} \left[ \sum_{u=\sigma(n-\tau)}^{s-1} \frac{1}{a(u)} \left( \sum_{v=u}^{s-1} \frac{1}{b(v)} \right) \right] \sum_{i=1}^{m} q_i(s) > 1,
\]

(5.2.2)

where \( \sigma(n-\tau) = \max \{\sigma_1(n-\tau), \sigma_2(n-\tau), ..., \sigma_m(n-\tau)\} \). Then every solution of equation (5.1.1) is oscillatory.

Proof. Let \( x(n) \) be a non-oscillatory solution of (5.1.1). Assume that \( x(n) \) is eventually positive. Since \( \sigma_i(n - \tau) \to \infty \) as \( n \to \infty \), \( i = 0, 1, 2, ..., m \), there exists \( n_1 \geq n_0 \) such that \( x(n) > 0 \) and \( x(\sigma_i(n - \tau)) > 0 \) for \( n \geq n_1 \). From (5.1.1) we have,

\[
\Delta (b(n)\Delta (a(n)\Delta x(n))) = -\sum_{i=1}^{m} q_i(n) f(x(\sigma_i(n - \tau))).
\]

(5.2.3)
Since \( q_i(n) \) is not negative and \( x(\sigma_i(n - \tau)) \) is positive for \( n \geq n_1 \), the right
hand side becomes non-positive, therefore, we have \( \Delta (b(n) \Delta (a(n) \Delta x(n))) \leq 0 \)
for \( n \geq n_1 \).

Thus \( x(n) \), \( \Delta x(n) \) and \( \Delta (a(n) \Delta x(n)) \) are monotone eventually one signed.
Now we want to show that there is a \( n_2 \geq n_1 \) such that for \( n \geq n_2 \),
\[
\Delta (a(n) \Delta x(n)) > 0. \tag{5.2.4}
\]
Suppose this not true, then \( \Delta (a(n) \Delta x(n)) \leq 0 \). Since \( q_i(n) \), \( i = 0, 1, 2, \ldots, m \) are
not identically zero and \( b(n) > 0 \), it is clear that there is \( n_3 \geq n_2 \) such that
\( b(n_3) \Delta (a(n_3) \Delta x(n_3)) \leq 0 \) then for \( n \geq n_3 \), we have
\[
b(n) \Delta (a(n) \Delta x(n)) \leq b(n_3) \Delta (a(n_3) \Delta x(n_3)) < 0. \tag{5.2.5}
\]
Dividing (5.2.5) by \( b(n) \) and then summing from \( n_3 \) to \( (n - 1) \), we obtain
\[
a(n) \Delta x(n) - a(n_3) \Delta x(n_3) < b(n_3) \Delta (a(n_3) \Delta x(n_3)) \sum_{s=n_3}^{n-1} \frac{1}{b(s)}. \tag{5.2.6}
\]
Letting \( n \to \infty \) in (5.2.6), and because of (H2), we see that \( a(n) \Delta x(n) \to -\infty \) as \( n \to \infty \). Thus there is \( n_4 \geq n_3 \) such that \( a(n_4) \Delta x(n_4) < 0 \), using
\( \Delta (a(n) \Delta x(n)) \leq 0 \), we have for \( n \geq n_4 \) that
\[
a(n) \Delta x(n) \leq a(n_4) \Delta x(n_4). \tag{5.2.7}
\]
If we divide (5.2.7) by \( a(n) \) and summing from \( n_4 \) to \( (n - 1) \) with \( n \to \infty \),
the right hand side becomes negative. Thus, we have \( x(n) \to -\infty \). But this
is a contradiction to \( x(n) \) being eventually positive and therefore it proves that
(5.2.4) hold. Now we consider the two cases. Suppose \( \Delta x(n) \) is eventually
positive, say $\Delta x(n) > 0$, for $n \geq n_2$. Define the function $z(n)$ by $z(n) = \sum_{i=1}^{m} \frac{\rho(n)}{x(\sigma_i(n - \tau))} b(n) \Delta (a(n) \Delta x(n))$. It is obvious that $z(n) > 0$ for $n \geq n_2$ and $\Delta z(n)$ is

$$
\Delta z(n) = \frac{\rho(n)}{\sum_{i=1}^{m} x(\sigma_i(n - \tau))} \Delta (b(n) \Delta (a(n) \Delta x(n))) + b(n + 1) \Delta (a(n + 1) \Delta x(n + 1)) \Delta \left( \frac{\rho(n)}{\sum_{i=1}^{m} x(\sigma_i(n - \tau))} \right),
$$

$$
\Delta z(n) = -\rho(n) \sum_{i=1}^{m} q_i(n) x(\sigma_i(n - \tau)) \left( \sum_{i=1}^{m} x(\sigma_i(n - \tau)) \right) \sum_{i=1}^{m} x(\sigma_i(n + 1 - \tau))
$$

That is,

$$
\Delta z(n) \leq -\rho(n) q(n) + z(n + 1) - \frac{\rho(n)}{\rho(n + 1)} \left( \frac{\sum_{i=1}^{m} x(\sigma_i(n + 1 - \tau))}{\sum_{i=1}^{m} x(\sigma_i(n - \tau))} \right) z(n + 1),
$$

(5.2.8)

where $q(n) = \min \{q_1(n), q_2(n), ..., q_m(n)\}$. On the other hand, using $\Delta (b(n) \Delta (a(n) \Delta x(n))) \leq 0$, $\Delta b(n) \geq 0$ and (5.2.4), we can find that

$$
\Delta (a(n) \Delta x(n)) \leq 0.
$$

(5.2.9)
Using (5.2.9), the equality
\[
a(n)\Delta x(n) = a(N)\Delta x(N) + \sum_{s=N}^{n-1} \Delta (a(s)\Delta x(s)),
\]
becomes,
\[
a(n)\Delta x(n) \geq (n - N)\Delta (a(n)\Delta x(n)), \text{ for } N \geq n_2.
\]
(5.2.11)

Since \(\Delta (a(n)\Delta x(n))\) is non-increasing, we obtain
\[
\Delta x(\sigma_i(n - \tau)) \geq \frac{(\sigma_i(n - \tau) - N) \Delta (a(n)\Delta x(n))}{a(\sigma_i(n - \tau))}.
\]
(5.2.12)

Summing (5.2.12) from \(n_1\) to \((n + 1) - 1\), we have
\[
x(\sigma_i(n + 1 - \tau)) \geq x(\sigma_i(n - \tau)) + \sum_{s=n_1}^{n} \frac{(\sigma_i(s - \tau) - N) \Delta (a(s)\Delta x(s))}{a(\sigma_i(s - \tau))},
\]
which implies
\[
x(\sigma_i(n + 1 - \tau)) \geq (n - n_1)\frac{(\sigma_i(n - \tau) - N) \Delta (a(n)\Delta x(n))}{a(\sigma_i(n - \tau))}.
\]
(5.2.13)

That is,
\[
\sum_{i=1}^{m} x(\sigma_i(n + 1 - \tau)) \geq \sum_{i=1}^{m} (n - n_1)\frac{(\sigma_i(n - \tau) - N) \Delta (a(n)\Delta x(n))}{a(\sigma_i(n - \tau))},
\]
\[
\sum_{i=1}^{m} x(\sigma_i(n + 1 - \tau)) \geq \frac{n - n_1}{b(n)\rho(n)}\frac{b(n)\Delta (a(n)\Delta x(n))}{\rho(n)} \sum_{i=1}^{m} \frac{(\sigma_i(n - \tau) - N)}{a(\sigma_i(n - \tau))} \sum_{i=1}^{m} x(\sigma_i(n + 1 - \tau)).
\]

That is,
\[
\sum_{i=1}^{m} x(\sigma_i(n + 1 - \tau)) \geq \frac{n - n_1}{b(n)\rho(n)}z(n) \sum_{i=1}^{m} \frac{(\sigma_i(n - \tau) - N)}{a(\sigma_i(n - \tau))}.
\]
(5.2.13)
Then using (5.2.13) in (5.2.8), it follows that
\[
\Delta z(n) \leq -\rho(n)q(n) + z(n + 1) - \frac{(n - n_1)z^2(n + 1)\sum_{i=1}^{m} (\sigma_i(n - \tau) - N)}{a(\sigma_i(n - \tau))b(n)\rho(n + 1)}.
\]

That is,
\[
\Rightarrow \Delta z(n) \leq -\left(\frac{1}{2} \sqrt{\frac{b(n)\rho(n + 1)}{(n - n_1)\sum_{i=1}^{m} (\sigma_i(n - \tau) - N)} - \frac{(n - n_1)\sum_{i=1}^{m} (\sigma_i(n - \tau) - N)}{a(\sigma_i(n - \tau))b(n)\rho(n + 1)}} + \frac{1}{4}(n - n_1)\sum_{i=1}^{m} \frac{\sigma_i(n - \tau) - N}{a(\sigma_i(n - \tau))} - \rho(n)q(n)\right)^2.
\]

It follows that
\[
\Delta z(n) \leq - \left(\rho(n)q(n) - \frac{1}{4} \frac{b(n)\rho(n + 1)}{(n - n_1)\sum_{i=1}^{m} (\sigma_i(n - \tau) - N)}\right).
\]

Summing (5.2.14) from \(N\) to \((n - 1)\) and letting \(n \to \infty\), we see that \(\lim_{n \to \infty} z(n) = -\infty\). This contradicts \(z(n)\) being eventually positive. If \(\Delta x(n)\) is eventually negative, we summing (5.1.1) from \(n\) to \(\infty\) and hence \(\frac{b(n)}{b(n+1)}(\Delta (a(n)\Delta x(n))) > 0\), we have
\[
-b(n)(\Delta (a(n)\Delta x(n))) + \sum_{s=n}^{\infty} \sum_{i=1}^{m} q_i(s)x(\sigma_i(n - \tau)) \leq 0.
\]

Now summing (5.2.15) from \(n\) to \(\infty\) after dividing by \(b(n)\) and using \(a(n)\Delta x(n) < 0\), will lead to
\[
a(n)\Delta x(n) + \sum_{s=n}^{\infty} \left(\sum_{u=n}^{s-1} \frac{1}{b(u)}\right) \sum_{i=1}^{m} q_i(s)x(\sigma_i(n - \tau)) \leq 0.
\]
Dividing (5.2.16) by \(a(n)\) and summing again from \(n\) to \(\infty\) gives
\[
\sum_{s=n}^{\infty} \left[ \sum_{u=n}^{s-1} \frac{1}{a(u)} \left( \sum_{v=u}^{s-1} \frac{1}{b(v)} \right) \right] \sum_{i=1}^{m} q_i(s) x(\sigma_i(n - \tau)) \leq x(n).
\] (5.2.17)

Replace \(n\) by \(\sigma(n - \tau)\) in (5.2.17), where \(\sigma(n - \tau) = \max \{\sigma_1(n - \tau), \sigma_2(n - \tau), ..., \sigma_m(n - \tau)\}\) will give
\[
\sum_{s=\sigma(n - \tau)}^{\infty} \left[ \sum_{u=\sigma(n - \tau)}^{s-1} \frac{1}{a(u)} \left( \sum_{v=u}^{s-1} \frac{1}{b(v)} \right) \right] \sum_{i=1}^{m} q_i(s) x(\sigma_i(n - \tau)) \leq x(\sigma(n - \tau)).
\] (5.2.18)

Using the fact that \(\sigma_i(n - \tau) < n\) and \(x(n)\) is decreasing in (5.2.18), we obtain
\[
\sum_{s=\sigma(n - \tau)}^{\infty} \left[ \sum_{u=\sigma(n - \tau)}^{s-1} \frac{1}{a(u)} \left( \sum_{v=u}^{s-1} \frac{1}{b(v)} \right) \right] \sum_{i=1}^{m} q_i(s) \leq 1.
\]

This is a contradiction to (5.2.2). Therefore, the proof is complete.

**Theorem 5.2.2.** Let \(h(n) = 0\) and \((H4)\) hold. Suppose that there exists a sequence \(\rho(n) > 0\) such that
\[
\sum_{s=n_2}^{\infty} \left( K_2 \rho(s) q(s) - \frac{b(s) \rho(s + 1)}{4(s - n_1) \sum_{i=1}^{m} \left( \sigma_i(s - \tau) - N \right) / \left( a(\sigma_i(s - \tau)) \right)} \right) = \infty,
\] (5.2.19)

where \(q(n) = \min\{q_1(n), q_2(n), ..., q_m(n)\}\) for every \(N \geq 0\), and that
\[
\lim_{n \to \infty} \max \sum_{s=\sigma(n - \tau)}^{n-1} \left[ \sum_{u=\sigma(n - \tau)}^{s-1} \frac{1}{a(u)} \left( \sum_{v=u}^{s-1} \frac{1}{b(v)} \right) \right] \sum_{i=1}^{m} q_i(s) = \infty,
\] (5.2.20)

where \(\sigma(n - \tau) = \max \{\sigma_1(n - \tau), \sigma_2(n - \tau), ..., \sigma_m(n - \tau)\}\). Then every solution of equation (5.1.1) is oscillatory.

**Proof.** The beginning part of the proof is similar to the proof of Theorem 5.2.1 until we reach to the existence of two possible cases. Suppose \(\Delta x(n)\) is eventually
positive, then we define \( z(n) \) by

\[
z(n) = \frac{\rho(n)}{\sum_{i=1}^{m} f(x(\sigma_i(n - \tau)))} b(n) \Delta (a(n) \Delta x(n)) > 0.
\]

0. It is obvious that \( z(n) > 0 \) for \( n \geq n_2 \) and \( \Delta z(n) \) is

\[
\Delta z(n) = \frac{\rho(n)}{\sum_{i=1}^{m} f(x(\sigma_i(n - \tau)))} \Delta (b(n) \Delta (a(n) \Delta x(n)))
\]

\[
+ b(n + 1) \Delta (a(n + 1) \Delta x(n + 1)) \Delta \left( \frac{\rho(n)}{\sum_{i=1}^{m} f(x(\sigma_i(n - \tau)))} \right).
\]

This implies,

\[
\Delta z(n) = -\rho(n) \frac{\sum_{i=1}^{m} q_i(n) f(x(\sigma_i(n - \tau)))}{\sum_{i=1}^{m} f(x(\sigma_i(n - \tau)))} \Delta (z(n + 1) - \frac{\rho(n)}{\rho(n + 1)} \left( \sum_{i=1}^{m} f(x(\sigma_i(n + 1 - \tau))) \right) z(n + 1). \tag{5.2.21}
\]

In view of (H4), (5.2.21) becomes

\[
\Delta z(n) \leq -K_2 \rho(n) q(n) + z(n + 1) - \frac{\rho(n)}{\rho(n + 1)} \left( \sum_{i=1}^{m} x(\sigma_i(n + 1 - \tau)) \right) z(n + 1), \tag{5.2.22}
\]

where \( q(n) = \min \{q_1(n), q_2(n), ..., q_m(n)\} \). Then using (5.2.13) in (5.2.22), we get

\[
\Delta z(n) \leq - \left( K_2 \rho(n) q(n) - \frac{1}{4} \frac{b(n) \rho(n + 1)}{(n - n_1) \sum_{i=1}^{m} (\sigma_i(n - \tau) - \tau)} \right). \tag{5.2.23}
\]
Summing (5.2.23) from $N$ to $(n-1)$ and letting $n \to \infty$, we see that $\lim_{n \to \infty} z(n) = -\infty$. This contradicts $z(n)$ being eventually positive. If $\Delta x(n)$ is eventually negative and proceeding as in the proof of Theorem 5.2.1 we will end up with

$$
\sum_{s=\sigma(n-\tau)}^{\infty} \left[ \sum_{u=\sigma(n-\tau)}^{s-1} \frac{1}{a(u)} \left( \sum_{v=u}^{s-1} \frac{1}{b(v)} \right) \right] \sum_{i=1}^{m} q_i(s) f \left( x(\sigma_i(n-\tau)) \right) \leq x(\sigma(n-\tau)),
$$

(5.2.24)

where $\sigma(n-\tau) = \max \{ \sigma_1(n-\tau), \sigma_2(n-\tau), ..., \sigma_m(n-\tau) \}$. Using the fact that $\sigma_i(n-\tau) < n$, $f(x)$ is increasing and $x(n)$ is decreasing in (5.2.24), we obtain

$$
\sum_{s=\sigma(n-\tau)}^{\infty} \left[ \sum_{u=\sigma(n-\tau)}^{s-1} \frac{1}{a(u)} \left( \sum_{v=u}^{s-1} \frac{1}{b(v)} \right) \right] \sum_{i=1}^{m} q_i(s) \leq \frac{x(\sigma(n-\tau))}{f(x(\sigma(n-\tau)))}, \text{ and then}
$$

$$
\lim_{n \to \infty} \sup_{s=\sigma(n-\tau)}^{\infty} \left[ \sum_{u=\sigma(n-\tau)}^{s-1} \frac{1}{a(u)} \left( \sum_{v=u}^{s-1} \frac{1}{b(v)} \right) \right] \sum_{i=1}^{m} q_i(s) \leq \lim_{n \to \infty} \sup_{s=\sigma(n-\tau)}^{\infty} \frac{1}{f(x(\sigma(n-\tau)))} = \frac{1}{K_1}.
$$

This is a contradiction to (5.2.20) and the proof is complete. \(\square\)

### 5.3 Examples

**Example 5.3.1.** Consider the difference equation

$$
\Delta ((n+1)\Delta (n\Delta x(n))) + \sum_{i=1}^{2} q_i(n)x(\sigma_i(n-\tau)) = 0, \ n \geq 12. \tag{E.5.1}
$$

Here $a(n) = n$, $b(n) = n + 1$, $q_1(n) = 9n^2 + 21n + 20$, $q_2(n) = n^2 + n + 6$, $\sigma_1(n) = n - 4$, $\sigma_2(n) = n - 9$, $\sigma_1(n-2) = n - 6$, $\sigma_2(n-2) = n - 11$, $\tau = 2$, $\rho_n = \frac{1}{n^2}$, $\sum_{n=12}^{\infty} \frac{1}{a(n)} = \sum_{n=12}^{\infty} \frac{1}{n} = \infty$, $\sum_{n=12}^{\infty} \frac{1}{b(n)} = \sum_{n=12}^{\infty} \frac{1}{n+1} = \infty$. \(70\)
\[
\sum_{i=1}^{2} q_i(n) = (10n^2 + 22n + 26),
\]

\[
\sum_{n=12}^{\infty} \left( \sum_{n=12}^{\infty} \frac{1}{a(n)} \sum_{n=12}^{\infty} \frac{1}{b(n)} \right) \sum_{i=1}^{2} q_i(n) = \sum_{n=12}^{\infty} \left( \sum_{n=12}^{\infty} \frac{1}{a(n)} \sum_{n=12}^{\infty} \frac{1}{b(n)} \right) (10n^2 + 22n + 26) = \infty
\]

and

\[
\sum_{n=12}^{\infty} \left( \rho(n)q(n) - \frac{b(n)\rho(n+1)}{4(n-n_1)} \sum_{i=1}^{2} \left( \sigma_i(n-2) - N \right) \frac{1}{a(\sigma_i(n-2))} \right) = \sum_{n=12}^{\infty} \left( \frac{n^2 + n + 6}{n^3} - \frac{(n-6)}{4(n+1)^2(2n^2 - 58n + 336)} \right) = \infty.
\]

All the conditions of Theorem 5.2.1 are satisfied. Hence equation (E.5.1) is oscillatory. In fact \(x(n)=(-1)^n\) is a solution of equation (E.5.1).

**Example 5.3.2.** Consider the difference equation

\[
\Delta ((n+1)\Delta (n\Delta x(n))) + \sum_{i=1}^{2} q_i(n)x^3(\sigma_i(n-\tau)) = 0, \ n \geq 6. \quad (E.5.2)
\]

Here \(a(n) = n\), \(b(n) = n + 1\), \(q_1(n) = 3n^2 + 18n + 4\), \(q_2(n) = 11n^2 + 38n + 4\),

\(\sigma_1(n) = n - 2\), \(\sigma_2(n) = n - 1\), \(\sigma_1(n-3) = n - 5\), \(\sigma_2(n-3) = n - 4\), \(\tau = 3\),

\(\rho_n = \frac{1}{n^3}, \ \sum_{n=6}^{\infty} \frac{1}{a(n)} = \sum_{n=6}^{\infty} \frac{1}{n} = \infty, \ \sum_{n=6}^{\infty} \frac{1}{b(n)} = \sum_{n=6}^{\infty} \frac{1}{n+1} = \infty, \ \sum_{i=1}^{2} q_i(n) = (14n^2 + 56n + 18),\)

\[
\sum_{n=12}^{\infty} \left( \sum_{n=12}^{\infty} \frac{1}{a(n)} \sum_{n=12}^{\infty} \frac{1}{b(n)} \right) \sum_{i=1}^{2} q_i(n) = \sum_{n=12}^{\infty} \left( \sum_{n=12}^{\infty} \frac{1}{a(n)} \sum_{n=12}^{\infty} \frac{1}{b(n)} \right) (14n^2 + 56n + 18) = \infty
\]
and

\[
\sum_{n=6}^{\infty} \left( K_2 \rho(n) q(n) - \frac{b(n) \rho(n+1)}{4(n - n_1)^2 \sum_{i=1}^2 \frac{(\sigma_i(n - 2) - N)}{a(\sigma_i(n - 2))}} \right)
\]

\[
= \sum_{n=6}^{\infty} \left( \frac{3n^2 + 18n + 4}{n^3} - \frac{(n - 4)}{4(n + 1)^2(2n^2 - 30n + 94)} \right) = \infty.
\]

All the conditions of Theorem 5.2.2 are satisfied. Hence equation (E.5.2) is oscillatory. In fact \(x(n)=(-1)^n\) is a solution of equation (E.5.1).