Chapter 4

OSCILLATION OF THIRD ORDER NONLINEAR DIFFERENCE EQUATIONS

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4.1 Introduction

In this chapter, we are concerned with the oscillation properties of all the solutions of a third order non-linear difference equation of the form

$$\Delta \left( \frac{1}{r_{n-1}} \Delta \left( \frac{1}{a_{n-1}} (\Delta y_{n-1})^\alpha \right) \right) - \frac{1}{p_{n-1}} (\Delta y_{n-1})^\alpha + \frac{1}{q_n} f(y_n) = 0, \quad n \in N,$$

(4.1.1)

where the following conditions are assumed to be hold.

(H1) \( \{r_n\}, \{a_n\}, \{p_n\} \) and \( \{q_n\} \) are real positive sequences and \( q_n \neq 0 \) for infinitely many values of \( n \).

(H2) \( f : R \to R \) is continuous and \( xf(x) > 0 \) for all \( x \neq 0 \).
(H3) there exists a real valued function \( g \) such that \( f(u) - f(v) = g(u, v)(u - v)^\beta, \)
for all \( u \neq 0 \) and \( v \neq 0 \) and \( g(u, v) \geq L > 0 \in R. \)

(H4) \( \{\phi_n\} \) is real positive sequence.

(H5) \( \sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty. \)

(H6) \( \sum_{n=n_0}^{\infty} \frac{r_n}{\phi_n} = \infty. \)

(H7) \( \sum_{n=n_0}^{\infty} \frac{\phi_n}{q_n} = \infty. \)

(H8) \( \frac{\phi_{n+1}}{p_n} \geq \frac{\Delta \phi_n}{a_n r_n}. \)

(H9) \( \alpha, \beta \) are positive ratios of odd integers.

(H10) \( \lim_{n \to \infty} \inf \sum_{s=0}^{n} \frac{\phi_s}{q_s} > -\infty. \)

(H11) \( \sum_{n=n_0}^{\infty} \frac{\phi_n}{p_n^2 r_n^2 a_n} < \infty. \)

(H12) \( \sum_{n=n_0}^{\infty} \frac{(\Delta \phi_n)^2}{r_n^2 a_n^2} < \infty. \)

(H13) \( \sum_{n=n_0}^{\infty} \left( \frac{\phi_{n+1}}{p_n} - \frac{\Delta \phi_n}{a_n r_n} \right)^2 \frac{1}{p_n \phi_n} < \infty. \)

The technique followed in this chapter is similar to the technique used in the book of Wong and Agarwal [65].

4.2 Basic Lemmas

Lemma 4.2.1. Let the function \( K(n, s, y) : N_{n_0} \times N_{n_0} \times R \to R \) be such that for each fixed \( n \) and \( s \), the function \( K(n, s, y) \) is non-decreasing in \( x. \)
Furthermore, let \( \{h_n\} \) be a given sequence and the sequences \( \{u_n\} \) and \( \{v_n\} \) be defined on \( N \) satisfying, for all \( n \in N_{n_0} \),

\[
    u_n \geq (\leq) h_n + \sum_{s=n_0}^{n-1} K(n, s, u_s), \tag{4.2.1}
\]

and

\[
    v_n = h_n + \sum_{s=n_0}^{n-1} P(n, s, v_s) \tag{4.2.2}
\]

respectively. Then \( u_n \geq (\leq) v_n \) for all \( n \in N_{n_0} \).

**Proof.** The proof can be found in [65]. \( \square \)

**Lemma 4.2.2.** Suppose that \( y_n > 0 (\leq 0) \) is a solution of (4.1.1) for \( n \in N_{n_0}^\gamma \) \((1 \leq n_0 \leq \gamma)\) and there exists a positive sequence \( \{\phi_n\} \), and \( m > 0 \) such that

\[
    - \Delta \left( \frac{1}{a_{n_0-1}} (\Delta y_{n_0})^\alpha \right) \phi_{n_0-1} - \frac{\Delta \left( \frac{1}{a_s} (\Delta y_s)^\alpha \right) \phi_s (\Delta y_s)^\beta g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s} + \frac{n}{s=n_0} \left( \frac{\phi_s}{q_s} - \frac{(\Delta y_{s-1})^\alpha \phi_s}{f(y_s)p_{s-1}} - \frac{\Delta \left( \frac{1}{a_{s-1}} (\Delta y_{s-1})^\alpha \right) (\Delta \phi_{s-1})}{f(y_s)r_{s-1}} \right) \geq m, \tag{4.2.3}
\]

for all \( n \in N_{n_0}^\gamma \). Then

\[
    \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \phi_n \geq (\leq) -mf(y_n)r_n, \quad n \in N_{n_0}^\gamma. \tag{4.2.4}
\]

**Proof.** Let \( z_n = \frac{1}{r_n} \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \phi_n \), then

\[
    \Delta z_n = \Delta \left( \frac{1}{r_n} \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \right) \phi_{n+1} + \frac{1}{r_n} \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) (\Delta \phi_n).
\]

Now,

\[
    \frac{\Delta z_{n-1}}{f(y_n)} = -\frac{\phi_n}{q_n} + \frac{(\Delta y_{n-1})^\alpha \phi_n}{f(y_n)p_{n-1}} + \frac{\Delta \left( \frac{1}{a_{n-1}} (\Delta y_{n-1})^\alpha \right) (\Delta \phi_{n-1})}{f(y_n)r_{n-1}},
\]

for all \( n \in N_{n_0}^\gamma \).
summing from \( n_0 \) to \( n \), where \( n \in N^\gamma_{n_0} \), we have

\[
-z_n \frac{f(y_{n+1})}{f(y_n)} = -z_{n_0-1} \frac{f(y_{n_0})}{f(y_n)} + \sum_{s=n_0}^{n} \left( \frac{\phi_s}{q_s} - \frac{(\Delta y_{s-1})^\alpha \phi_s}{f(y_s)p_{s-1}} - \Delta \left( \frac{1}{a_{s-1}} (\Delta y_{s-1})^\alpha \phi_{s-1} \right) \frac{f(y_s)r_{s-1}}{f(y_s)f(y_{s+1})} \right) \\
+ \sum_{s=n_0}^{n} \Delta \left( \frac{1}{a_x}(\Delta y_x)^\alpha \right) \frac{\phi_x (\Delta y_x)^\beta g(y_{x+1}, y_x)}{f(y_x)f(y_{x+1})r_x}. \tag{4.2.5}
\]

In view of (4.2.3), we see further that

\[
-z_n \geq m f(y_{n+1}) + \sum_{s=n_1}^{n} \Delta \left( \frac{1}{a_x}(\Delta y_x)^\alpha \right) \frac{\phi_x (\Delta y_x)^\beta g(y_{x+1}, y_x)}{f(y_x)f(y_{x+1})r_x}. \tag{4.2.6}
\]

**CASE 1**: Suppose that \( y_n > 0 \). Then (4.2.6) implies \( -z_n > 0 \), or equivalently \( \Delta y_n < 0 \), \( n \in N^\gamma_{n_1} \). Let \( u_n = -z_n = -\frac{1}{r_n} \Delta \left( \frac{1}{a_n}(\Delta y_n)^\alpha \right) \phi_n \). Then (4.2.6) becomes

\[
u_n \geq m f(y_{n+1}) + \sum_{s=n_1}^{n} \frac{f(y_{n+1})(-\Delta y_n)^\beta g(y_{n+1}, y_n) u_s}{f(y_s)f(y_{s+1})}. \tag{4.2.7}
\]

also let \( v_n = m f(y_{n+1}) + \sum_{s=n_1}^{n} \frac{f(y_{n+1})(-\Delta y_n)^\beta g(y_{n+1}, y_n) v_s}{f(y_s)f(y_{s+1})} \). \tag{4.2.8}

Using Lemma 4.2.1, we have from (4.2.7) and (4.2.8), that

\[
u_n \geq v_n. \tag{4.2.9}
\]

From (4.2.8), we find

\[
u_n \frac{f(y_{n+1})}{f(y_{n+1})} = \Delta \left( m + \sum_{s=n_1}^{n} \frac{(-\Delta y_n)^\beta g(y_{n+1}, y_n)}{f(y_s)f(y_{s+1})} v_s \right) \\
= \frac{(-\Delta y_{n+1})^\beta g(y_{n+2}, y_{n+1})}{f(y_{n+1})f(y_{n+2})} v_{n+1}. \tag{4.2.10}
\]

On the other hand

\[
u_n \frac{f(y_{n+1})}{f(y_{n+1})} = \frac{\Delta v_n}{f(y_{n+1})} - \frac{(-\Delta y_{n+1})^\beta g(y_{n+2}, y_{n+1})}{f(y_{n+1})f(y_{n+2})} v_{n+1}. \tag{4.2.11}
\]
Equating (4.2.10) and (4.2.11), we obtain

\[ \Delta v_n = 0 \text{ and so } v_n = v_{n1} = mf(y_{n1}), \ n \in N_{n1}^\gamma. \tag{4.2.12} \]

From (4.2.9) and (4.2.12), we obtain

\[ \Delta \left( \frac{1}{a_n} (\Delta y_n)^a \right) \phi_n \leq -mf(y_{n1})r_n. \]

**CASE 2:** Suppose that \( y_n < 0 \). Then (4.2.6) gives \( z_n > 0 \), or equivalently \( \Delta y_n > 0, \ n \in N_{n1}^\gamma \). Let \( u_n = z_n = \frac{1}{r_n} \Delta \left( \frac{1}{a_n} (\Delta y_n)^a \right) \phi_n \). It follows from (4.2.6) that

\[ u_n \geq -mf(y_{n+1}) + \sum_{s=n1}^{n} \frac{[-f(y_{n+1})](\Delta y_s)^3 g(y_s + 1, y_s)}{f(y_s)f(y_{s+1})} u_s, \]

also let \( v_n = -mf(y_{n+1}) + \sum_{s=n1}^{n} \frac{[-f(y_{n+1})](\Delta y_s)^3 g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})} v_s \).

As in **CASE 1**, \( \Delta v_n = 0 \) and hence \( v_n = v_{n1} = -mf(y_{n1}), \ n \in N_{n1}^\gamma \). Then inequality (4.2.9) immediately reduces to from (4.2.4). The proof is complete.

\[ \Box \]

**Corollary 4.2.1.** Let \( \{y_n\} \) be a positive solution of (4.1.1) and there exists a positive sequence \( \{\phi_n\} \) such that (H5), (H6) and (H10) hold, then

\[ \sum_{s=n1}^{\infty} \frac{\Delta \left( \frac{1}{a_s} (\Delta y_s)^a \right) \phi_s(\Delta y_s)^3 g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s} < \infty. \tag{4.2.13} \]

**Proof.** Suppose

\[ \sum_{s=n1}^{\infty} \frac{\Delta \left( \frac{1}{a_s} (\Delta y_s)^a \right) \phi_s(\Delta y_s)^3 g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s} = \infty, \]

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hence there exists $n_1^* \geq n_1$, such that
\[ \Delta \left( \frac{1}{a_{n_0}} (\Delta y_{n_0})^\alpha \right) \phi_{n_0 - 1} f(y_{n_0}) r_{n_0 - 1} + \sum_{s=n_0}^{n} \frac{\phi_s}{q_s} + \sum_{s=n_0}^{n_1^* - 1} \Delta \left( \frac{1}{a_s} (\Delta y_s)^\alpha \right) \phi_s (\Delta y_s)^\beta g(y_{s+1}, y_s) \frac{f(y_s) f(y_{s+1}) r_s}{f(y_s) f(y_{s+1}) r_s} \geq m, \]

where $m > 0$ is a constant, Lemma 4.2.2 implies that
\[ \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \leq -m f(y_{n_1^*}) \frac{r_n}{\phi_n} \text{ for } n \geq n_1^*. \] (4.2.15)

Summing the last inequality from $n_2$ to $(n - 1)$, we have
\[ \frac{1}{a_n} (\Delta y_n)^\alpha \leq \frac{1}{a_{n_2}} (\Delta y_{n_2})^\alpha - m f(y_{n_1^*}) \sum_{s=n_2}^{n-1} \frac{r_s}{\phi_s}. \]

In view of (H6), $\frac{1}{a_n} (\Delta y_n)^\alpha \to -\infty$ as $n \to \infty$, i.e., $\frac{1}{a_n} (\Delta y_n)^\alpha < -k_1$, $k_1 > 0$, $\Delta y_n < -k_1 a_n^\frac{1}{\alpha}$. Summing the last inequality from $n_3$ to $(n - 1)$, we have
\[ y_n < y_{n_3} - k_1^\frac{1}{\alpha} \sum_{s=n_3}^{n-1} a_s^\frac{1}{\alpha}, \text{ for } n \geq n_1^*. \] (4.2.16)

In view of (H5), the relation (4.2.16) implies that $y_n$ is negative eventually, which is a contradiction. The proof is complete. \(\square\)

**Lemma 4.2.3.** Let $\{\phi_n\}$ be a positive sequence. Suppose that

(i) $\lim_{|y| \to \infty} |f(y)| = \infty$,

(ii) $\lim_{n \to \infty} \sum_{s=n_0}^{n} \frac{\phi_s}{q_s}$ exists.

Let $y_n$ be a non-oscillatory solution of (4.1.1), then
\[ \sum_{s=n_0}^{\infty} \Delta \left( \frac{1}{a_s} (\Delta y_s)^\alpha \right) \phi_s (\Delta y_s)^\beta g(y_{s+1}, y_s) \frac{f(y_s) f(y_{s+1}) r_s}{f(y_s) f(y_{s+1}) r_s} < \infty. \] (4.2.17)
\[
\lim_{n \to \infty} \frac{\Delta \left( \frac{1}{a_n}(\Delta y_n)^\alpha \right) \phi_n}{f(y_{n+1})r_n} = 0, \tag{4.2.18}
\]

and
\[
\frac{\Delta \left( \frac{1}{a_n}(\Delta y_n)^\alpha \right) \phi_n}{f(y_{n+1})r_n} = \sum_{s=n+1}^\infty \frac{\phi_s}{q_s} + \sum_{s=n+1}^\infty \frac{\Delta \left( \frac{1}{a_s}(\Delta y_s)^\alpha \right) \phi_s(\Delta y_s)^\beta g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s}, \tag{4.2.19}
\]

for sufficiently large \( n \).

**Proof.** Let \( \{y_n\} \) be a non-oscillatory solution of (4.1.1). Without loss of generality, assume that \( y_n > 0 \) for \( n \geq n_0 \). By corollary 4.2.1, it follows that (4.2.17) holds. Similar to the proof of Lemma 4.2.2, it follows that (4.2.5) holds. We rewrite (4.2.5) as

\[
\frac{z_n}{f(y_{n+1})} - \frac{z_{n-1}}{f(y_n)} = \sum_{s=n_0}^\infty \frac{\phi_s}{q_s} - \sum_{s=n_0}^\infty \frac{\Delta \left( \frac{1}{a_s}(\Delta y_s)^\alpha \right) \phi_s(\Delta y_s)^\beta g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s} + \sum_{s=n_0}^\infty \frac{\phi_s}{q_s} + \sum_{s=n_0}^n \frac{(\Delta y_{s-1})^\alpha \phi_s}{f(y_s)p_{s-1}} + \sum_{s=n_0}^n \frac{(\Delta y_{s-1})^\alpha (\Delta \phi_{s-1})}{f(y_s)r_{s-1}}, \tag{4.2.20}
\]

this implies

\[
\frac{z_n}{f(y_{n+1})} = \eta + \sum_{s=n+1}^\infty \frac{\phi_s}{q_s} + \sum_{s=n+1}^\infty \frac{\Delta \left( \frac{1}{a_s}(\Delta y_s)^\alpha \right) \phi_s(\Delta y_s)^\beta g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s} + \sum_{s=n_0}^n \frac{(\Delta y_{s-1})^\alpha \phi_s}{f(y_s)p_{s-1}} + \sum_{s=n_0}^n \frac{(\Delta \phi_{s-1})}{f(y_s)r_{s-1}},
\]

where

\[
\eta = \frac{z_{n_0-1}}{f(y_{n_0})} - \sum_{s=n_0}^\infty \frac{\phi_s}{q_s} - \sum_{s=n_0}^\infty \frac{\Delta \left( \frac{1}{a_s}(\Delta y_s)^\alpha \right) \phi_s(\Delta y_s)^\beta g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s}. \tag{4.2.21}
\]
We claim that $\eta = 0$. If $\eta < 0$, we choose $n_2$ sufficiently large so that
\[ \left| \sum_{s=n_2}^{\infty} \frac{\phi_s}{q_s} \right| \leq -\frac{\eta}{4}, \quad n \geq n_2, \]
and
\[ \sum_{s=n_2}^{\infty} \Delta \left( \frac{1}{a_s} (\Delta y_s)^{\alpha} \right) \phi_s (\Delta y_s)^{\beta} g(y_{s+1}, y_s) \frac{1}{f(y_s)f(y_{s+1})r_s} < -\frac{\eta}{4}, \]
we take $n_0 = n_1 = n_2$ in Lemma 4.2.2, so that all assumption of Lemma 4.2.2 hold. From Lemma 4.2.2 and (4.2.16), we obtain
\[ \Delta y_n < -k_1^\frac{1}{\alpha} a_1^\frac{1}{\alpha}, \quad \text{for } n \geq n_2, \]
which yields a contradiction to the fact that $y_n > 0$ since (H5) holds. If $\eta > 0$, from (4.2.20), we have
\[ \Delta \left( \frac{1}{a_n} (\Delta y_n)^{\alpha} \right) \phi_n \frac{1}{f(y_{n+1})r_n} = \eta > 0, \]
which implies that $\Delta \left( \frac{1}{a_n} (\Delta y_n)^{\alpha} \right) > 0$, eventually. Summing the above inequality from $n$ to $\infty$, we obtain
\[ \Delta y_n < -m^\frac{1}{\alpha} a_m^\frac{1}{\alpha}, \quad m > 0, \]
which contradicts the positivity of $\{y_n\}$ since (H5) holds. Hence the proof of the lemma is complete. \qed

4.3 Results Related to Oscillation Behavior

**Theorem 4.3.1.** Suppose that $g(u,v) \geq L > 0$ for $u \neq v$ and there exists a positive sequence $\{\phi_n\}$ such that (H5), (H6), (H7), (H8), (H11) and (H12) hold, then equation (4.1.1) is oscillatory.
Proof. To the contrary, let $y_n$ be a non-oscillatory solution of (4.1.1), which may (and do) assume to be eventually positive. That is, $y_n > 0$, for $n \geq M - 1$. For the sake of convenience, let $w_n = \frac{1}{r_n} \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \phi_n$, for $n \geq M$. Then

$$\Delta w_n = \Delta \left( \frac{1}{r_n} \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \right) \phi_{n+1} + \frac{1}{r_n} \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) (\Delta \phi_n),$$

which implies

$$\frac{\Delta w_n}{f(y_{n+1})} = -\frac{\phi_{n+1}}{q_{n+1}} + \frac{(\Delta y_n)^\alpha \phi_{n+1}}{f(y_{n+1})p_n} + \frac{\Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right)}{f(y_{n+1})r_n} (\Delta \phi_n).$$

Since

$$\Delta \left( \frac{w_n}{f(y_{n+1})} \right) = \frac{\Delta w_n}{f(y_{n+1})} - \frac{w_n g(y_{n+1}; y_n)(\Delta y_n)^\beta}{f(y_n)f(y_{n+1})},$$

Summing the above equation from $M$ to $(n - 1)$, we have

$$\frac{\Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \phi_n}{f(y_n)r_n} + \sum_{s=M}^{n-1} \frac{(\Delta y_s)^\alpha \phi_{s+1}}{f(y_{s+1})p_s} + \frac{(\Delta y_s)^\alpha (\Delta \phi_s)}{f(y_{s+1})a_s r_s} - \frac{(\Delta y_{s+1})^\alpha (\Delta \phi_s)}{f(y_{s+1})a_{s+1} r_s} \right)$$

$$+ \sum_{s=M}^{n-1} \frac{(\Delta y_s)^\alpha \phi_s (\Delta y_s)^\beta g(y_{s+1}; y_s)}{f(y_s)f(y_{s+1})r_s} = \frac{\Delta \left( \frac{1}{a_M} (\Delta y_M)^\alpha \right) \phi_M}{f(y_M)r_M}.$$ (4.3.1)

Using Schwarz’s inequality, we have

$$\sum_{s=M}^{n-1} \frac{(\Delta y_s)^\alpha \phi_{s+1}}{f(y_{s+1})p_s} \leq \left( \sum_{s=M}^{n-1} \frac{\phi_{s+1}}{p_s^2 r_s^2 a_s} \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{a_s r_s^2 \phi_{s+1} (\Delta y_s)^{2\alpha}}{f^2(y_{s+1})} \right)^{\frac{1}{2}},$$ (4.3.2)

$$\sum_{s=M}^{n-1} \frac{(\Delta y_s)^\alpha (\Delta \phi_s)}{f(y_{s+1})a_s r_s} \leq \left( \sum_{s=M}^{n-1} \frac{(\Delta \phi_s)^2}{a_s^2 r_s^2} \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{(\Delta y_s)^{2\alpha}}{f^2(y_{s+1})} \right)^{\frac{1}{2}},$$ (4.3.3)

$$\sum_{s=M}^{n-1} \frac{(\Delta y_{s+1})^\alpha (\Delta \phi_s)}{f(y_{s+1})a_{s+1} r_s} \leq \left( \sum_{s=M}^{n-1} \frac{(\Delta \phi_s)^2}{a_{s+1}^2 r_s^2} \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{(\Delta y_{s+1})^{2\alpha}}{f^2(y_{s+1})} \right)^{\frac{1}{2}}.$$ (4.3.4)
Now, we use $g(u, v) \geq L > 0$ for $u \neq v$, (4.3.2), (4.3.3) and (4.3.4) in (4.3.1) to get

$$\frac{\Delta \left( \frac{1}{a_n} (\Delta y_n)^{\alpha} \right)}{f(y_n)} \phi_n + \sum_{s=M}^{n-1} \phi_{s+1} + L \sum_{s=M}^{n-1} \frac{\Delta \left( \frac{1}{a_n} (\Delta y_s)^{\alpha} \right) \phi_s (\Delta y_s)^\beta}{f(y_s) f(y_{s+1}) r_s}
$$

$$- \left( \sum_{s=M}^{n-1} \phi_{s+1} \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{a_s r_s^2 \phi_{s+1} (\Delta y_s)^{2\alpha}}{f^2(y_{s+1})} \right)^{\frac{1}{2}} + \left( \sum_{s=M}^{n-1} \frac{(\Delta \phi_s)^2}{a_s^2 r_s^2} \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{(\Delta y_s)^{2\alpha}}{f^2(y_{s+1})} \right)^{\frac{1}{2}}
$$

$$- \left( \sum_{s=M}^{n-1} \frac{(\Delta \phi_s)^2}{a_s^2 r_s^2} \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{(\Delta y_{s+1})^{2\alpha}}{f^2(y_{s+1})} \right)^{\frac{1}{2}} \leq \frac{\Delta \left( \frac{1}{a_M} (\Delta y_M)^{\alpha} \right) \phi_M}{f(y_M) r_M}.$$  \hspace{1cm} (4.3.5)

Note that

$$L \sum_{s=M}^{n-1} \frac{\Delta \left( \frac{1}{a_n} (\Delta y_s)^{\alpha} \right) \phi_s (\Delta y_s)^\beta}{f(y_s) f(y_{s+1}) r_s}
$$

$$+ \left( \sum_{s=M}^{n-1} \frac{(\Delta \phi_s)^2}{a_s^2 r_s^2} \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{(\Delta y_s)^{2\alpha}}{f^2(y_{s+1})} \right)^{\frac{1}{2}} \leq \frac{\Delta \left( \frac{1}{a_M} (\Delta y_M)^{\alpha} \right) \phi_M}{f(y_M) r_M}.$$  \hspace{1cm} (4.3.6)

remains bounded below as $n \to \infty$. Thus, taking (H7) into account, we observe from (4.3.5) that $\frac{\Delta \left( \frac{1}{a_n} (\Delta y_n)^{\alpha} \right) \phi_n}{f(y_n) r_n} \to -\infty$ as $n \to \infty$. Hence there exists an integer $M_1 \geq M$ such that $\Delta \left( \frac{1}{a_n} (\Delta y_n)^{\alpha} \right) < 0$, for $n \geq M_1$. That is, $\Delta \left( \frac{1}{a_n} (\Delta y_n)^{\alpha} \right) < -k_2$, $k_2 > 0$, summing the last inequality from $m_1$ to $(n-1)$, we have $\frac{1}{a_n} (\Delta y_n)^{\alpha} < \frac{1}{a_{m_1}} (\Delta y_{m_1})^{\alpha} - k_2 (n-m_1)$. Therefore $\frac{1}{a_n} (\Delta y_n)^{\alpha} \to -\infty$ as $n \to \infty$, hence there exists $M_2 \geq M_1$, such that

$$\Delta y_n < 0, \text{ for } n \geq M_2.$$  \hspace{1cm} (4.3.6)
We rewrite (4.3.1) as

\[
\frac{\Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \phi_n}{f(y_n)r_n} + \sum_{s=M_2}^{n-1} \frac{\Delta \left( \frac{1}{a_s} (\Delta y_s)^\alpha \right) \phi_s (\Delta y_s)^\beta g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s} = \frac{\Delta \left( \frac{1}{a_M} (\Delta y_M)^\alpha \right) \phi_M}{f(y_M)r_M} - \sum_{s=M}^{M_2-1} \frac{\Delta \left( \frac{1}{a_s} (\Delta y_s)^\alpha \right) \phi_s (\Delta y_s)^\beta g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s}
\]

\[
- \sum_{s=M}^{M_2-1} \frac{\Delta (\Delta y_{s+1})^\alpha (\Delta \phi_s)}{f(y_{s+1})a_{s+1}r_s} + \sum_{s=M}^{M_2-1} \left( \frac{\phi_{s+1}}{p_s} - \frac{\Delta \phi_s}{a_sr_s} \right) \frac{(\Delta y_s)^\alpha}{f(y_{s+1})}
\]

and use (H8), (4.3.1) and (4.3.6) to find as an integer \( M_3 \geq M_2 \) such that

\[
\frac{\Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \phi_n}{f(y_n)r_n} + \sum_{s=M_2}^{n-1} \frac{\Delta \left( \frac{1}{a_s} (\Delta y_s)^\alpha \right) \phi_s (\Delta y_s)^\beta g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s} \leq -k_3, \ n \geq M_3
\]

where \( k_3 \) is a positive constant. Hence

\[
- \frac{\Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \phi_n}{f(y_n)r_n} - \sum_{s=M_2}^{n-1} \frac{\Delta \left( \frac{1}{a_s} (\Delta y_s)^\alpha \right) \phi_s (\Delta y_s)^\beta g(y_{s+1}, y_s)}{f(y_s)f(y_{s+1})r_s} \geq k_3. \quad (4.3.7)
\]

Let \( u_n = -\frac{1}{r_n} \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \phi_n \). Then (4.3.7) becomes

\[
u_n \geq k_3 f(y_n) + \sum_{s=M_3}^{n} \frac{f(y_n)(- (\Delta y_s)^\beta g(y_{s+1}, y_s))}{f(y_s)f(y_{s+1})} u_s, \ n \geq M_3, \quad (4.3.8)\]

also let \( v_n = k_3 f(y_n) + \sum_{s=M_3}^{n} \frac{f(y_n)(- (\Delta y_s)^\beta g(y_{s+1}, y_s))}{f(y_s)f(y_{s+1})} v_s \).

(4.3.9)

Using Lemma 4.2.1, from (4.3.8) and (4.3.9) we have,

\[
u_n \geq u_n. \quad (4.3.10)\]

Dividing (4.3.9) by \( f(y_n) \) and then applying the forward difference operator \( \Delta \), it is easy to verify that \( \Delta v_n \equiv 0 \). Therefore

\[
u_n \geq v_n = k_3 f(y_{M_3}), \ n \geq M_3,
\]

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this implies

\[ \Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right) \leq -k_3 f(y_{M_3}) \frac{r_n}{\phi_n}. \]

Summing the last inequality from \( m_2 \) to \( (n - 1) \), we have

\[ \frac{1}{a_n} (\Delta y_n)^\alpha \leq \frac{1}{a_{m_2}} (\Delta y_{m_2})^\alpha - k_3 f(y_{M_3}) \sum_{s=m_2}^{n-1} \frac{r_s}{\phi_s}. \]

In view of (H6), \( \frac{1}{a_n} (\Delta y_n)^\alpha \to -\infty \), as \( n \to \infty \). That is, \( \Delta y_n < -k_4 a_n^{1/\alpha} \).

Summing the last inequality from \( M_3 \) to \( (n - 1) \), we obtain

\[ y_n < y_{M_3} - k_4^{1/\alpha} \sum_{s=M_3}^{n-1} a_s^{1/\alpha}. \]

In view of (H5), we have \( y_n \to -\infty \) as \( n \to \infty \), which yields a contradiction to the fact that \( \{y_n\} \) is eventually positive. The proof is similar to the case when \( \{y_n\} \) is eventually negative. This completes the proof.

**Theorem 4.3.2.** Suppose that

(i) \( 0 < \int_{\epsilon}^{\infty} \frac{dy}{f(y)^{1/\alpha}} < \infty \), \( \int_{-\infty}^{-\epsilon} \frac{dy}{f(y)^{1/\alpha}} < \infty \), for every \( \epsilon > 0 \),

(ii) \( \sum_{s=n_0}^{\infty} \phi_s/q_s \) and \( \sum_{s=n_0}^{\infty} r_s/\phi_s \) exists, and

\[ \lim_{n \to \infty} \sum_{s=n_0}^{n} a_{s+1}^{1/\alpha} \left( \sum_{l=n_1}^{n} \frac{r_l}{\phi_l} \sum_{l=n+1}^{\infty} \frac{\phi_l}{q_l} \right)^{1/\alpha} = \infty, \quad (4.3.11) \]

then equation (4.1.1) is oscillatory.

**Proof.** Suppose to the contrary. Without loss of generality, we assume that (4.1.1) has an eventually positive solution. Under our assumption, Lemma 4.2.3 is true. Let \( \{y_n\} \) be an eventually positive solution of (4.1.1), then (4.2.13) hold.
Since $f$ is non-decreasing and the second sum in (4.2.19) is non-negative, here
\[
\frac{\Delta \left( \frac{1}{a_n} (\Delta y_n)^\alpha \right)}{f(y_{n+1})} \geq \frac{r_n}{\phi_n} \sum_{s=n+1}^{\infty} \phi_s.
\]
Summing it from $n_1$ to $n$, we obtain
\[
\frac{\Delta y_{n+1}}{f(y_{n+1})^\frac{1}{\alpha}} > a_{n+1} \left( \sum_{s=n_1}^{n} \frac{r_t}{\phi_t} \sum_{l=n+1}^{\infty} \frac{\phi_l}{l} \right)^{\frac{1}{\alpha}}.
\]
Summing it from $n_0$ to $n$, we have
\[
\sum_{s=n_0}^{n} \frac{\Delta y_{n+1}}{f(y_{n+1})^\frac{1}{\alpha}} > \sum_{s=n_0}^{n} a_{s+1} \left( \sum_{t=n_1}^{n} \frac{r_t}{\phi_t} \sum_{l=n+1}^{\infty} \frac{\phi_l}{l} \right)^{\frac{1}{\alpha}}.
\]
We define $w(t) = y_{n+1} + (t - n) \Delta y_{n+1}$, $n + 1 \leq t \leq n + 2$. If $\Delta y_{n+1} > 0$, then $y_{n+1} \leq w(t) \leq y_{n+2}$; if $\Delta y_{n+1} < 0$, then $y_{n+2} \leq w(t) \leq y_{n+1}$ and $dw = \Delta y_{n+1}$.

Let $H(y) = \int_y^\infty \frac{dw}{f(w(t))}^\frac{1}{\alpha}$,
then $H(y_{n_0}) = \int_{w(n_0)}^{\infty} \frac{dy}{f(y)^\frac{1}{\alpha}} \geq \int_{n_0}^{n+1} \frac{dr}{f(w(t))^\frac{1}{\alpha}} \geq \sum_{s=n_0}^{n} \frac{\Delta y_{s+1}}{f(y_{s+1})^\frac{1}{\alpha}}$
\[
\geq \sum_{s=n_0}^{n} a_{s+1} \left( \sum_{t=n_1}^{n} \frac{r_t}{\phi_t} \sum_{l=n+1}^{\infty} \frac{\phi_l}{l} \right)^{\frac{1}{\alpha}},
\]
which contradicts (4.3.11). Similarly, we can prove that (4.1.1) does not posses an eventually negative solution. The proof is complete.

\textbf{Theorem 4.3.3.} Suppose that $g(u, v) \geq L > 0$ for $u \neq v$ and there exists a positive sequence $\{\phi_n\}$ such that (H5), (H6), (H7), (H8), (H12) and (H13) hold. Then equation (4.1.1) is oscillatory.

\textbf{Proof.} We proceed as in the proof of Theorem 3.1 to obtain (3.1). Then we use Schwarz’s inequality, to get
\[
\sum_{s=M}^{n-1} \left( \frac{\phi_{s+1}}{p_s} - \frac{\Delta \phi_s}{a_s r_s} \right) (\Delta y_s)^\alpha \leq \left( \sum_{s=M}^{n-1} \left( \frac{\phi_{s+1}}{p_s} - \frac{\Delta \phi_s}{a_s r_s} \right)^2 \frac{1}{p_s \phi_s} \right)^{\frac{1}{2}} \left( \sum_{s=M}^{n-1} \frac{p_s \phi_s (\Delta y_s)^{2\alpha}}{f^2(y_{s+1})} \right)^{\frac{1}{2}},
\]
and (4.3.4). Hence, an inequality similar to (4.3.5) holds. The rest of the proof follows Theorem 4.3.1. The proof is complete.

4.4 Examples

Example 4.4.1. Consider the difference equation

\[
\Delta \left( \frac{1}{n+1} \Delta \left( \frac{(n-1)^2}{n} (\Delta y_{n-1})^{\frac{2}{3}} \right) \right) - \frac{1}{n-1} (\Delta y_{n-1})^{\frac{2}{3}} + \frac{(n+2)(n+4)}{n} \left( y_n^5 + \frac{y_n}{3} \right) = 0,
\]

(E.4.1)

\( n \geq 2 \) and \( \phi_n = \frac{n+1}{n+2} \). Here \( r_n = n+2 \), \( a_n = \frac{n+1}{n^2} \), \( p_n = n \), \( q_n = \frac{n}{n+1} \).

\[
\frac{\phi_n}{r_n} = \frac{n}{(n+1)(n+4)}, \quad \sum_{n=2}^{\infty} a_n^\frac{3}{2} = \sum_{n=2}^{\infty} \left( \frac{n+1}{n^2} \right)^{\frac{3}{2}} = \infty, \quad \sum_{n=2}^{\infty} \frac{\phi_n}{r_n} = \sum_{n=2}^{\infty} \frac{(n+2)^2}{n+1} = \infty, \quad \sum_{n=2}^{\infty} \frac{\phi_n}{p_n} = \sum_{n=2}^{\infty} \frac{1}{(n+2)^3} < \infty, \quad \sum_{n=2}^{\infty} \frac{(\Delta \phi_n)^2}{p_n^2 a_n^2} = \sum_{n=2}^{\infty} \frac{n^4}{(n+1)^2(n+2)^2(n+3)^2} < \infty.
\]

All the conditions of Theorem 4.3.1 are satisfied. Hence equation (E.4.1) is oscillatory.

Example 4.4.2. Consider the difference equation

\[
\Delta \left( \frac{(n-1)n(n+1)^2(n+2)}{12} \Delta \left( \frac{n(n+1)}{n-1} (\Delta y_{n-1})^{\frac{2}{3}} \right) \right) - \frac{1}{3(n-1)} (\Delta y_{n-1})^{\frac{2}{3}} + \frac{(n^2+3n+2)}{n+3} \left( y_n^5 + \frac{y_n}{3} \right) = 0,
\]

(E.4.2)

\( n \geq 2 \) and \( \phi_n = \frac{6}{n(n+1)(n+2)} \). Here \( r_n = \frac{12}{n(n+1)(n+2)^2(n+3)} \), \( a_n = \frac{n}{(n+1)(n+2)}, \quad p_n = 3n, \quad q_n = \frac{n+3}{n^2+3n+2} \).
\[ \sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty} \left( \frac{n}{(n+1)(n+2)} \right)^2 = \infty, \quad \sum_{n=2}^{\infty} r_n = \sum_{n=2}^{\infty} \frac{2}{(n+2)(n+3)} < \infty, \]
\[ \sum_{n=2}^{\infty} \phi_n q_n = \sum_{n=2}^{\infty} \frac{6}{n(n+3)} < \infty. \] All the conditions of Theorem 4.3.2 are satisfied.

Hence equation (E.4.2) is oscillatory.

**Example 4.4.3.** Consider the difference equation

\[ \Delta \left( \frac{1}{n+1} \Delta \left( \frac{n^2}{n+1} (\Delta y_{n-1}) \right) \right) - \frac{1}{n+1} (\Delta y_{n-1}) \]
\[ + \frac{n^2}{n+1} y_n^7 = 0, \quad (E.4.3) \]

\( n \geq 2 \) and \( \phi_n = \frac{n}{n+1} \). Here \( r_n = n+2 \), \( a_n = \frac{n+2}{(n+1)^2} \), \( p_n = n+2 \), \( q_n = \frac{n+1}{n^2} \), \( \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{n+2}{(n+1)^2} = \infty, \)
\[ \sum_{n=2}^{\infty} r_n = \sum_{n=2}^{\infty} \frac{(n+1)(n+2)}{n} = \infty, \]
\[ \sum_{n=2}^{\infty} \phi_n q_n = \sum_{n=2}^{\infty} \frac{n^3}{(n+1)^2} = \infty, \]
\[ \frac{\phi_{n+1}}{p_n} \geq \frac{\Delta \phi_n}{a_n r_n} \Rightarrow (n+2) \geq 1, \]
\[ \sum_{n=2}^{\infty} \left( \frac{\Delta \phi_n}{p_n} \right)^2 a_n^2 = \sum_{n=2}^{\infty} \frac{(n+1)^2}{n(n+2)^6} < \infty, \]
\[ \sum_{n=2}^{\infty} \left( \frac{\phi_{n+1}}{p_n} - \frac{\Delta \phi_n}{a_n r_n} \right) \frac{1}{p_n \phi_n} = \sum_{n=2}^{\infty} \frac{(n+1)^3}{n(n+2)^4} < \infty. \] All the conditions of Theorem 4.3.3 are satisfied. Hence equation (E.4.3) is oscillatory.