Chapter 10

Three-dimensional modulational instability of magnetized electron-acoustic solitary waves

10.1 Introduction

Nonlinear behavior of natural phenomena is one of the most important aspect of modern plasma physics. Of particular interest in the study of nonlinear waves, the excitation, propagation and interaction of solitary waves is an important issue in theoretical physics. Among the best known paradigms used to study nonlinear behavior are different versions of Korteweg-de Vries (KdV) equation, or nonlinear Schrödinger equation (NLSE). Some form of reductive perturbation theory (RPT) is used to derive these equations. The KdV equation describes the evolution of the unmodulated wave and the bare pulse does not contain high frequency oscillations inside the packet. This special solution is also called KdV soliton, in which dispersion is compensated by the nonlinearity. On the other hand, NLSE describes the slow modulation of a monochromatic plane wave. For a medium with a positive coefficient of the cubic nonlinearity term in NLSE, the instability takes place in
direction transverse to the propagation and is called self focusing whereas that one in the longitudinal direction is referred to as the modulational instability. Even when a modulation takes place in a direction oblique to carrier wave propagation, the instability is still termed as a modulational instability. Here, the nonlinearities are in balance with wave group dispersion and the resulting solutions of NLSE possess envelope structures, known as envelope solitons. For an envelope soliton there has been a great deal of interest in studying the modulational instability (MI) of different wave modes in plasma because of its importance in stable wave propagation (Esfandyari-Kalejahi and Asgari (2005)).

The amplitude modulation (AM) of waves propagating in nonlinear dispersive media is a unique nonlinear phenomena that is relevant to many areas of physics and technology. This modulation which manifests itself as slow spatio and temporal variation of wave’s amplitude, may be due to several nonlinear mechanisms as parametric wave coupling, interaction between high and low-frequency modes or self-interaction of carrier waves. As far as plasma modes are concerned, the AM and MI have theoretically been investigated in different fluid models of plasmas (Mohan and Buti (1979); Bai-Song (2001); El-Labany et al. (2003a); Kourakis and Shukla (2003); Kourakis and Shukla (2004); Kourakis and Shukla (2005)). Such theoretical predictions related to nonlinear wave propagation have also been confirmed by experiments (Watanabe (1975); Bailung and Nakamura (1993); Luo et al. (1998); Nakamura et al. (1999); Nakamura and Sarma (2001)). Earlier (Hasegawa (1970;1972)) and recent (Eliasson and Shukla (2004)) numerical simulations also predict such behavior.

Electron-acoustic (EA) wave, an electrostatic wave which had been earlier discovered experimentally (Derfler and Simonen (1969); Henry and Treumann
10.1. Introduction

(1972); Ikezawa and Nakamura (1981)) in a plasma where a minority of inertial cold electrons oscillate against a dominant thermalized background of inertialess hot electrons providing the necessary restoring force. The wave is analogous to ion-acoustic wave but ion dynamics here plays no role except charge neutralization. The mode is restrictive in the sense that it demands \(T_c \ll T_h\) and further the hot electron population should represent a significant fraction of total electron density. Various satellite missions encourage two-electron-temperature plasmas in space. The localized potential variation in different regions of space plasma where co-existence of electron populations is encountered, have been reported by several spacecraft missions, e.g., the FAST at the auroral region (Delory et al. (1998); Ergun et al. (1998); Pottelette et al. (1999); McFadden et al. (2003)), as well as the S3-3 (Temerin (1982)), Viking (Bostrom (1988)), GEOTAIL, and POLAR (Matsumoto et al. (1994); Franz et al. (1998); Cattell et al. (1999); McFadden et al. (2003)). Also it has been shown that localized electrostatic structures are of nonlinear type and solitary waves in plasma sheet boundary layers are an electron waves, possibly an electron acoustic solitary waves (EASWs). From a nonlinear plasma-theoretical point of view, several investigations on EA waves have been reported in the recent past (Singh and Lakhina (2001); Eliasson and Shukla (2004); Singh and Lakhina (2004)). The relevance of EASWs considered more crucial, particularly, in Earth’s bow shock in the upstream region, EASWs have been suggested as the possible source of broadband electrostatic noise (BEN). They are also of potential importance in interpreting BEN observed in cusp of terrestrial magnetosphere space in auroral region and geomagnetic tail (Tokar and Gray (1984); Schrifer and Asour-Abdalla (1989); Berthomier et al. (2000); Singh and Lakhina (2001)). However, most of the models used in the above investigation
of EASWs are based on KdV or Sagdeev pseudopotential model. McFadden et al. (2003) has suggested that neither the velocity dependence of the observed potential structure amplitudes nor their asymmetry should be taken for granted, since they may be attributed to intrinsic measurement errors. Furthermore, the observed phase speeds lie over an extended region of values. These facts seem to suggest that employing KdV model may not be helpful for elucidation of the generation of the soliton structures and an alternative instability mechanisms may be present. In this context, modulational instability of different plasma wave modes have been studied. Specially some recent investigations on MI of EAWs have been reported to interpret the observational data (Kourakis and Shukla (2003); Kourakis and Shukla (2004); Kourakis and Shukla (2005)). Such investigations are also supported by the observation of localized modulated wave packets, encountered in abundance, e.g, in the Earth’s magnetosphere (Pottelette et al. (1999); Alpert (2001); Santolik et al. (2003)). However, the most of the investigations reported are limited to derive NLSE in one dimension.

Recent trends in the field of nonlinear science is growing for study of the behavior of nonlinear wave in multidimensions. Some earlier investigations reported in this perspective include, e.g., Davey-Stewartson (DS) equations, the Kadomtsev-Petviashvili (KP) equation, Zakharov-Kuznetsov (ZK) equation (Duan (2003); Xue (2004b)) etc. However only a few problems, in higher dimension are analytically integrable. Moreover, the multidimensional problems of nonlinear behavior are yet not well understood. Higher dimensional modulational instability for various plasma modes in different plasma systems has been studied recently by some investigators (Duan (2003); Duan (2004); Xue (2004b); Xue (2005b); Duan (2006); Zhang et al. (2006)). DS equation is shallow water limit of the Benney-
Roskes equation and have been studied extensively in many fields of nonlinear physics (Schults et al. (1987); Boiti et al. (1988); Fokas and Santini (1989); Fokas and Santini (1990); Pang et al. (1990); Ablowitz and Clarkson (1991); Nishinari et al. (1993)). The DS equation is an isotropic Lax integrable extension of the well known (1+1) dimensional NLSE. Modulational instability and periodic solitons of the DS equation have been investigated by Tajiri et al. (2002). However, above investigations are limited to unmagnetized plasmas. The modulation of magnetized electron acoustic solitary waves in higher dimensions has not been considered. In this part of the thesis, modulation of three-dimensional (3D) electron acoustic solitary waves in a magnetized plasma is studied. Using multiple scale method, a three-dimensional nonlinear Schrödinger equation (3D NLSE) in magnetized plasma is derived. The modulational properties of electron-acoustic (EA) waves and its stability regions in three-dimensional plasma have been studied in detail.

10.2 Governing Equations

Since the plasma with two electron populations do occur frequently in laboratory and space, EA waves play an important role in such environment. We consider a collisionless infinite homogeneous and magnetized plasma in a following model. The plasma fluid model consist of cold and hot electron components referred to here c and h respectively. The presence of two such population group is necessary for the existence of EA wave. The unnormalized basic equations governing the dynamics of electron acoustic wave are given as follow:

\[ \frac{\partial n_c}{\partial t} + \nabla (n_c V_c) = 0 \] (10.1)
where $\nabla = \left( \partial/\partial x, \partial/\partial y, \partial/\partial z \right)$, $V_c = (u_c, v_c, w_c)$ is the electron fluid velocity with $u_c$, $v_c$, $w_c$ as its components along the $x$, $y$ and $z$ directions respectively. $n_c$, $n_h$ and $\phi$ represents the cold and hot electron number density and electrostatic wave potential. The variables $t$, $(x, y, z)$, $n_c$, $(u_c, v_c, w_c)$ and $\phi$ are normalized by the reciprocal electron plasma frequency $\omega_{pe}^{-1} = \sqrt{(m/4\pi n_c_0 e^2)}$, hot electron Debye length $\lambda_{Dh} = \sqrt{(K_B T_h/4\pi n_h_0 e^2)}$, unperturbed equilibrium cold (hot) electron density $n_{c0}$($n_{h0}$), effective electron fluid speed $c_e = \sqrt{(K_B T_h/\alpha m)}$ and $K_B T_h / e$ respectively. Here $n_h = (1 - \beta \phi + \beta \phi^2)e^\phi$, $\theta = T_h/T_c$, $\alpha = n_{h0}/n_{c0}$ and $\omega_c = e B_0 / m c n_{c0} \omega_{pe}$ is the normalized electron gyrofrequency. Here $m$ is the electron mass, $e$ is the magnitude of the electron charge and $K_B$ is the Boltzmann’s constant. The dimensionless basic equations governing the dynamics of electron acoustic waves are given as follow:

$$\frac{\partial n_c}{\partial t} + \nabla \cdot (n_c V_c) = 0$$  \hspace{1cm} (10.4)

$$\frac{\partial V_c}{\partial t} + V_c \cdot \nabla V_c = \alpha \nabla \phi - \frac{5\alpha(1+\alpha)^{2/3}}{3\theta} n_c^{-1/3} \nabla n_c + \omega_c V_c \times \hat{z}$$  \hspace{1cm} (10.5)

$$\nabla^2 \phi = \frac{n_c}{\alpha} + n_h - \left( 1 + \frac{1}{\alpha} \right)$$  \hspace{1cm} (10.6)

In order to investigate the modulation of weakly 3D EAWs in given plasma system, we employ the standard reductive perturbation technique to obtain the wave governing equations. Considering the strongly magnetized plasma and that the wave propagates in $z$ direction with weak transverse perturbations, the independent variables can be stretched as $\xi = \epsilon x$, $\eta = \epsilon y$, $\zeta = \epsilon (z - v_g t)$ and $\tau = \epsilon^2 t$, where $v_g$ is the group velocity to be determined by the compatibility requirement. Here $\epsilon$ is a small formal expansion parameter and is the measure
of perturbation. The condition $\epsilon \ll 1$ implies that the plasma dimension must be much larger than the Debye length, which is satisfied in the most cases of interest. We will assume that all perturbed quantities depend on the fast scale via the phase $\chi = kz - \omega t$ only, while the slow scales enter the argument of the $l$th harmonic amplitude, say for density as $n_l^n$. Following this prescription, the dependent variables are expanded as:

$$
\begin{pmatrix}
  n_c \\
  \phi \\
  u_c \\
  v_c \\
  w_c
\end{pmatrix}
= \begin{pmatrix}
  1 \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}
+ \sum_{n=1}^{\infty} \sum_{l=-\infty}^{\infty} \epsilon^n
\begin{pmatrix}
  n_l^n(\xi, \eta, \zeta, \tau) \\
  \phi_l^n(\xi, \eta, \zeta, \tau) \\
  e u_l^n(\xi, \eta, \zeta, \tau) \\
  e v_l^n(\xi, \eta, \zeta, \tau) \\
  w_l^n(\xi, \eta, \zeta, \tau)
\end{pmatrix}
\epsilon^{il(kz - \omega t)}
$$

(10.7)

where $n_c, u_c, v_c, w_c$ and $\phi$ satisfy the reality condition $A_{-l}^{(n)} = A_l^{n*}$ and the asterisk denotes the complex conjugate. One should note that the perpendicular velocity components $(u_c, v_c)$ in equation (10.7) appear at higher order in $\epsilon$ than does the parallel velocity component $w_c$. This anisotropy is introduced by the strong magnetic field. The implication is that the fluid gyromotion is treated as a higher order effect in this approximation.

Using stretching coordinates and substituting equation (10.7) into equations (10.4)-(10.6) and equating each coefficient of $\epsilon$, we obtain, at the order of $\epsilon$, the following equations for $|l| = 1$:

$$
w_1^1 = -\frac{\alpha \omega (1 - \beta + k^2)}{k} \phi_1^1
$$

(10.8)

$$
n_1^1 = -\alpha (1 - \beta + k^2) \phi_1^1
$$

(10.9)

$$
\omega^2 = \frac{k^2}{1 - \beta + k^2} + \frac{5k^2\alpha (1 + \alpha)^{2/3}}{3\theta}
$$

(10.10)

If $l > 1$, $w_l^1 = n_l^1 = \phi_l^1 = 0$. We also obtain, at the order of $\epsilon^2$ from equations (10.4), (10.5), for $l = 0$ and at the order of $\epsilon$ of (10.6), the following relation
For $l = 0$ from equations (10.4), (10.5) and (10.6), we obtain the group velocity

$$v_g = \frac{k}{\omega} \left[ \frac{1 - \beta}{(1 - \beta + k^2)^2} + \frac{5\alpha(1 + \alpha)^{2/3}}{3\theta} \right]$$  \hspace{1cm} (10.11)$$

We obtain the following evolution equations at the order of $\epsilon^2$ for $l = 2$ from equations (10.4), (10.5) and (10.6):

$$\begin{pmatrix} w_2^2 \\ \phi_2^2 \\ n_2^2 \end{pmatrix} = \begin{pmatrix} A_w \\ \tilde{A}_\phi \\ \tilde{A}_n \end{pmatrix} (\phi_1^1)^2$$  \hspace{1cm} (10.12)$$

where

$$\tilde{A}_w = \frac{\alpha \omega (1 - \beta + k^2)}{6k^3} \left[ \frac{\alpha (1 - \beta + k^2)(1 - \beta + 4k^2)}{9\theta} + 40\alpha^2(1 + \alpha)^{2/3}(1 - \beta + k^2)^2(1 - \beta + 4k^2) + 2\alpha(1 - \beta + k^2)^2 + 1 \right]$$  \hspace{1cm} (10.13)$$

$$\tilde{A}_\phi = \frac{-\frac{2}{\alpha}(kA_w + \alpha^2(1 - \beta + k^2)^2) - 1}{2(1 - \beta + 4k^2)}$$  \hspace{1cm} (10.14)$$

$$\tilde{A}_n = \frac{kA_w^2 \omega + \alpha^2(1 - \beta + k^2)^2}{\omega}$$  \hspace{1cm} (10.15)$$

We obtain the following equations at the order of $\epsilon^3$ for $l = 0$ from equations (10.4), (10.5) and the order of $\epsilon^2$ from equation (10.6):

$$\begin{pmatrix} n_0^2 \\ \phi_0^2 \\ w_0^2 \end{pmatrix} = \begin{pmatrix} \tilde{B}_n \\ \tilde{B}_\phi \\ \tilde{B}_w \end{pmatrix} |\phi_1^1|^2$$  \hspace{1cm} (10.16)$$

where

$$\tilde{B}_n = \frac{-\alpha}{1 - (1 - \beta)(v_g^2 - \frac{5\alpha(1 + \alpha)^{2/3}}{3\theta})} \left[ \frac{2\alpha(1 - \beta)(1 - \beta + k^2)^2}{\omega v_g + \frac{5\alpha(1 + \alpha)^{2/3}}{3\theta}} \right]$$  \hspace{1cm} (10.17)$$

$$+ \frac{\alpha(1 - \beta)(1 - \beta + k^2)^2 + 1}{k}$$
10.2. Governing Equations

\[ \tilde{B}_\phi = -\frac{1}{1-\beta} \left( \frac{\tilde{B}_n}{\alpha} + 1 \right) \] \hspace{2cm} (10.18)

\[ \tilde{B}_w = v_g \tilde{B}_n - \frac{2\omega\alpha^2(1-\beta+k^2)^2}{k} \] \hspace{2cm} (10.19)

Further, the equations at the order of \( \epsilon^3 \) for \( l = 1 \) from equations (10.4), (10.5), and (10.6) are obtained and eliminating the terms \( n_1^3, w_1^3 \) and \( \phi_1^3 \) and then we deduce NLSE for the EASWs as:

\[ \partial_t \phi + \tilde{P} \frac{\partial^2 \phi}{\partial \xi^2} + \tilde{Q}(|\phi|)^2 \phi + \tilde{R} \left( \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} \right) = 0 \] \hspace{2cm} (10.20)

\[ \tilde{P} = \frac{-3k^4}{2\omega^3(1-\beta+k^2)^2} \left[ (1-\beta) + \frac{(1-\beta)v_g\omega}{k} - \frac{(1-\beta)^2}{(1-\beta+k^2)^2} \right] \] \hspace{2cm} (10.21)

\[ \tilde{Q} = \tilde{Q}_0 + \tilde{Q}_1 + \tilde{Q}_2 \] \hspace{2cm} (10.22)

\[ \tilde{Q}_0 = \frac{k}{2\alpha\omega^2(1-\beta+k^2)^2} \left[ 3\omega k(1-\beta+k^2)^2 + \frac{10v_g\omega(1-\beta+k^2)^3}{3} \right. \\
- \frac{10\omega k(1-\beta)(1-\beta+k^2)}{3} - \frac{2\omega^2 v_g\alpha^3(1-\beta+k^2)^3}{9} + \frac{\omega k\alpha(1+3\beta)}{2(1-\beta+k^2)} \\
- \frac{k\alpha\omega}{(1-\beta)(1-\beta+k^2)} - \frac{k\omega(2\alpha(1-\beta+k^2)^2 - 1)}{2(1-\beta+k^2)(1-\beta+4k^2)} \right] \] \hspace{2cm} (10.23)

\[ \tilde{Q}_1 = \frac{k}{2\alpha\omega^2(1-\beta+k^2)^2} \left[ -k^2 A_w \left( \frac{1}{(1-\beta+k^2)(1-\beta+4k^2)} \right) \right. \\
+ \left. 3\alpha \left( 1 + \frac{8v_g\omega(1-\beta+k^2)}{9k} \right) - \frac{8(1-\beta)}{9(1-\beta+k^2)} \right] \] \hspace{2cm} (10.24)

\[ \tilde{Q}_2 = \frac{k}{2\alpha\omega^2(1-\beta+k^2)^2} \left[ -\tilde{B}_n k\omega \left( \alpha + \frac{1}{(1-\beta)(1-\beta+k^2)} \right) \right. \\
+ \frac{2\alpha(1-\beta)}{(1-\beta+k^2)} + \frac{8\alpha v_g\omega(1-\beta+k^2)}{3k} - \frac{8\alpha(1-\beta)}{3(1-\beta+k^2)} \right] \] \hspace{2cm} (10.25)
In the 3D NLSE (10.20), for notational convenience, we have replaced $\phi_1$ by $\phi$. In equation (10.20), $\tilde{P}$ and $\tilde{Q}$ are the dispersion and nonlinear coefficients respectively.

### 10.3 Modulational Instability of EASWs

Now, we analyze the modulational instability of EASWs described by 3D NLSE equation (10.20). The NLSE equation (10.20) has the trivial homogeneous solution

$$\phi = \phi_0 \exp(i \tilde{Q} | \phi_0 |^2 \tau)$$  \hspace{1cm} (10.27)

where $\phi_0$ is a real constant representing the amplitude of the carrier wave. Now we investigate the development of the small modulation $\delta \phi$ and $\delta \psi$ according to

$$\phi = [\phi_0 + \delta \phi(\xi, \eta, \zeta, \tau)] \exp(i \tilde{Q} | \phi_0 |^2 \tau)$$  \hspace{1cm} (10.28)

Substituting equation (10.28) into equation (10.20) and collecting terms in the first order (linearization), we obtain

$$i \frac{\partial \delta \phi}{\partial \tau} + \tilde{P} \frac{\partial^2 \delta \phi}{\partial \xi^2} + \tilde{Q}(| \phi_0 |^2(\delta \phi + \delta \phi^*) + \tilde{R} \left( \frac{\partial^2 \delta \phi}{\partial \xi^2} + \frac{\partial^2 \delta \phi}{\partial \eta^2} \right) = 0$$ \hspace{1cm} (10.29)

where $\delta \phi^*$ is the complex conjugate of $\delta \phi$. Letting $\delta \phi = U + iV$ and $(U,V) = (U_0,V_0) \exp \left[ i(K_x \xi + K_y \eta + K_z \zeta - \tilde{\Omega} \tau) \right] + (c.c.)$ in equation (10.29), and separating the real and imaginary parts, we finally obtain the following coupled equations:

$$i \tilde{\Omega} V_0 + \left[ 2 \tilde{Q} | \phi_0 |^2 - \tilde{P} K_z^2 + \tilde{R} (K_x^2 + K_y^2) \right] U_0 = 0$$ \hspace{1cm} (10.30)

$$-i \tilde{\Omega} U_0 - \left[ \tilde{P} K_z^2 - \tilde{R} (K_x^2 + K_y^2) \right] V_0 = 0$$ \hspace{1cm} (10.31)

where $K_x \xi + K_y \eta + K_z \zeta - \tilde{\Omega} \tau$ is the modulation phase with $K = (K_x^2 + K_y^2 + K_z^2)^{1/2}$ and $\tilde{\Omega}$ are, respectively, the wave number and the frequency of the modulation, and $K_x, K_y, K_z$ are the modulation wave numbers in x,y,z directions, respectively.

We
then obtain from equations (10.30) and (10.31) the following nonlinear dispersion relation for the amplitude modulation of the EASW modes:

\[ \tilde{\Omega}^2 = \left[ \hat{P} K_z^2 - \hat{R} (K_x^2 + K_y^2) \right]^2 \left[ 1 - \frac{2 |\phi_0|^2}{\hat{P} K_z^2 - \hat{R} (K_x^2 + K_y^2)} \hat{Q} \right] \]  

(10.32)

The modulational instability will be set in if the following condition is satisfied:

\[ \frac{2 |\phi_0|^2 \hat{Q}}{\hat{P} K_z^2 - \hat{R} (K_x^2 + K_y^2)} > 1 \]  

(10.33)

Letting \( K_x^2 + K_y^2 + K_z^2 = K^2 \) and \( \alpha' = \frac{K_z}{(K_x^2 + K_y^2)^{1/2}} \), equation (10.32) and the instability criterion (10.33) reduces to

\[ \tilde{\Omega}^2 = K^4 \left( \frac{\hat{P}\alpha'^2 - \hat{R}}{1 + \alpha'^2} \right)^2 \left[ 1 - \frac{2 |\phi_0|^2 (1 + \alpha'^2)}{K^2} \frac{\hat{Q}/\hat{P}}{\alpha'^2 - \hat{R}/\hat{P}} \right] \]  

(10.34)

\[ K^2 < K_c^2 = 2 |\phi_0|^2 (1 + \alpha'^2) \frac{\hat{Q}/\hat{P}}{\alpha'^2 - \hat{R}/\hat{P}} \]  

(10.35)

where \( \alpha' \) is related to the modulational obliqueness \( \tilde{\theta} \) according to \( \tilde{\theta} = \arctan(\alpha') \). So the modulation instability may occur if one of the following two conditions is verified:

\[ \hat{Q}\hat{P} > 0, \alpha'^2 > \hat{R}/\hat{P} \]  

(10.36)

or

\[ \hat{Q}\hat{P} < 0, \alpha'^2 < \hat{R}/\hat{P} \]  

(10.37)

Hence, the conditions (10.36) and (10.37) indicate that, if \( \omega_c > \omega \), the instability criterion is \( \hat{Q}\hat{P} > 0 \) for all obliqueness (\( \alpha' \)) because of \( \alpha'^2 - \hat{R}/\hat{P} > 0 \). When \( \omega_c < \omega \), however, there exists a critical value of obliqueness \( \tilde{\theta} \): \( \tilde{\theta}_c = \arctan(\hat{R}/\hat{P})^{1/2} \), as \( \tilde{\theta} < \tilde{\theta}_c \), the instability may be set in if \( \hat{P}\hat{Q} < 0 \) is satisfied. On the other hand, as \( \tilde{\theta} > \tilde{\theta}_c \), the instability may be set in when \( \hat{P}\hat{Q} > 0 \) is satisfied. That is, when the wave frequency is less than the cyclotron frequency, the instability criterion is simple and is similar to that in the unmagnetized 1D case. But when the wave
frequency is larger than the cyclotron frequency, the instability criterion is related
to modulational obliqueness, which gives much richer instability behavior. Figure
10.1(a-d) shows the instability regions distributed in the $\omega \sim \tilde{\theta}$ plane for different
$\beta$ at $\omega_c = 1.45$. It is clear that a critical value of $\omega = \omega_{cr}$ exists: $\tilde{P}\tilde{Q} > 0$ when
$\omega > \omega_{cr}$ and $\tilde{P}\tilde{Q} < 0$ when $\omega < \omega_{cr}$. The proceeding discussion shows that the
$\omega \sim \tilde{\theta}$ plane is divided into different parts with different stabilities by the line of
$\tilde{\theta} = \tilde{\theta}_c$, $\omega = \omega_c$ and $\omega = \omega_{cr}$. The value of $\omega_{cr}$ changes with the parameter $\beta$, hence,
the unstable region is modified significantly by $\beta$. When $\omega < \omega_c$, the instability
is independent of the obliqueness $\tilde{\theta}$. When $\omega > \omega_c$, however, the modulational
instability is related to the modulational obliqueness. For $\beta = 0.2$, the unstable
regions are distributed in two regions of $\omega_c < \omega < \omega_{cr} = 1.82$, $\tilde{\theta} < \tilde{\theta}_c$ and $\omega > \omega_{cr},$
$\tilde{\theta} > \tilde{\theta}_c$ (shown in Figure 10.1a). For $\beta = 0.25$ (see Figure 10.1b), the value of $\omega_{cr}$
decreases to $\omega_c$ i.e., $\omega_{cr} = \omega_c = 1.45$ and the unstable region is distributed in one
region of $\omega > \omega_c$, $\tilde{\theta} > \tilde{\theta}_c$. As $\beta$ further increases but is limited in the interval of
$0.25 < \beta < 0.35$ (for example see Figure 10.1c), the value of $\omega_{cr}$ is obtained in the
region of $0 < \omega_{cr} < \omega_c$ (for example, $\omega_{cr} = 1.1$ for $\beta = 0.3$ case). In this case, the
unstable regions are distributed in two parts of $\omega_{cr} < \omega < \omega_c$ for all obliqueness
and $\omega > \omega_c$, $\tilde{\theta} > \tilde{\theta}_c$. For $\beta = 0.35$ (Figure 10.1d), $\omega_{cr}$ decreases to $\omega_{cr} \approx 0$. On the
other hand, when $\beta > 0.35$, the value of $\omega_{cr}$ increases with $\beta$ and the changing
characters of the unstable regions shown in $0 < \beta < 0.35$ cases re-occurred (not
shown). For example, the instability characters in the $0.35 < \beta < 0.4$ case are
similar to that in the $0.25 < \beta < 0.35$ case, and that for the $\beta > 0.4$ case is similar
to that for the $0 < \beta < 0.25$ case. It should bear in mind that the unstable regions
will also be modified with the changing of $\omega_c$.

The maximum growth rate $\Gamma = \text{Im}(\tilde{\Omega})_{max}$ can be obtained from equation
10.3. Modulational Instability of EASWs

\[ \Theta^\prime \frac{\omega}{\Theta} \]

\[ \Theta^\prime \frac{\omega}{\Theta} \geq 0, \text{Stable} \]

\[ \Theta^\prime \frac{\omega}{\Theta} \leq 0, \text{Stable} \]

\[ \Theta^\prime \frac{\omega}{\Theta} \leq 0, \text{unStable} \]

\[ \Theta^\prime \frac{\omega}{\Theta} > 0, \text{unStable} \]
Figure 10.1: Unstable regions with $\omega_c = 1.45$, (a) $\beta = 0.2$, (b) $\beta = 0.25$, (c) $\beta = 0.3$ and (d) $\beta = 0.35$
10.3. Modulational Instability of EASWs

(10.34) according to \( \frac{\partial \tilde{\Omega}}{\partial K} = 0, \frac{\partial \tilde{\Omega}}{\partial \alpha} = 0 \), that result in

\[ K^2 = |\phi_0|^2 \tilde{Q} \left[ (1 + \alpha'^2) / (\tilde{P} \alpha'^2 - \tilde{R}) \right] \]

with the maximum growth rate \( \Gamma_m = Im(\tilde{\Omega}) = |\phi_0|^2 \tilde{Q} \). On the other hand, the maximum growth rate occurs when the condition

\[ \tilde{P} K_x^2 - \tilde{R} (K_x^2 + K_y^2) = |\phi_0|^2 \tilde{Q} \]

is satisfied. To glean more information about the nature of MI, we have plotted the growth rate \( \Gamma \) as a function of \( K \) for different \( \beta \), i.e., nonthermal parameter (figure 10.2), different \( \alpha (= \frac{nh}{n_e}) \) and different \( \theta (= \frac{T_h}{T_e}) \). The growths exhibit an increase with \( K \), attain maximum values followed by sharp decrease in all the three cases. The effects of increase of \( \alpha \) and \( \beta \) are to increase the instability growth rate with cut off at higher wave number shown in figure 10.2 and figure 10.3 respectively. However, increase of \( \theta \) leads to decrease the instability growth rate with cut off at lower wave number observed in figure 10.4.

Figure 10.2: Variation of growth rate \( \Gamma \) against \( K \) for different values of \( \beta \) with \( \alpha = 1.5 \) and \( \theta = 500 \).
Figure 10.3: Variation of growth rate $\Gamma$ against $K$ for different values of $\alpha(=\frac{n_{\text{eq}}}{n_{0\alpha}})$ with $\beta = 0.2$ and $\theta = 500$.

Figure 10.4: Variation of growth rate $\Gamma$ against $K$ for different values of $\theta(=\frac{T_{\text{th}}}{T_{c}})$ with $\beta = 0.2$ and $\alpha = 1.5$. 
10.4 Conclusions

It is known that the envelope electron acoustic (EA) nonlinear waves in one dimension are governed by the Nonlinear Schrödinger equation. In this investigation we have studied the modulation of three-dimensional (3D) electron acoustic solitary waves (EASW) in a magnetized plasma. Using reductive perturbation method, a three-dimensional nonlinear Schrödinger equation (3D NLSE) in magnetized plasma is obtained. The modulational properties of electron-acoustic (EA) wave and its stability regions in three-dimensional plasma has been studied. Unstable regions are modified with changing $\omega_c$. It is also noticed that the growth rate is significantly modified by the nonthermal parameter $\beta$, $\alpha$ and $\theta(=\frac{T_e}{T_i})$ respectively. The present work is representative of a class of nonlinear wave structures observed in auroral zone and some other regions of space plasmas.