Chapter 9

Dust-acoustic solitary waves in a dusty plasma in the presence of positrons

9.1 Introduction

The study of solitary waves (e.g. ion-acoustic waves, dust ion-acoustic waves and dust-acoustic waves) in plasma is very important not only from an academic point of view, but also from the view of its vital role in understanding the nonlinear features of localized electrostatic disturbances in space and laboratory environments. Washimi and Taniuti (1966) predicted theoretically the basic features of the solitary wave in electron-ion plasmas and later on these were verified experimentally (Ikezi et. al. (1970)). There has been a rapidly growing interest in the physics of dusty plasmas not only because dust is an ubiquitous component of universe, but also because of its vital role in understanding different collective processes in astrophysical and space environments. Classification and characterization of these collective modes are well documented in standard texts on dusty plasmas (Verheest (2000); Shukla and Mamun (2002)). The presence
of charged dust particles in plasma generate new modes, e.g., DIAWs, DAWs etc. These low frequency linear modes have been investigated particularly, dust ion-acoustic (DIA) waves (Shukla and Silin (1992); Shukla (1992)), dust-acoustic (DA) waves (Rao et al. (1990)) etc. These are the normal modes of unmagnetized dusty plasma. However, the linear wave theory based upon small amplitude approximation breaks down completely when amplitude becomes considerably large. Nonlinearity, inhomogeneity, dispersivity and dissipativity are the general features of dusty plasmas (Xue (2005a)). The nonlinearity effects in dusty plasmas contribute to the localization of waves, leading to different types of coherent nonlinear wave structures. In the research of collective effects involving highly charged dust grains in dusty plasmas, for example, the widely studied dust-acoustic waves (DAWs) (Rao et al. (1990); Barkan et al. (1995); Mamun et al. (1996a); Mamun et al. (1996b); Tagare (1997)), the dust charge is usually assumed as constant. Nejoh (1997b) pointed out that the dust grains have variable charge due to fragmentation, coalescence, and others. Since the dust charge variation with parameters such as electro- static plasma potential, electron and ion densities would influence the collective characteristics of the plasma, the effect of dust charge variation is of crucial importance in understanding dusty plasma waves. In the past, some authors (Ma and Liu (1997); Xie et al. (1998a); Xie et al. (1998b)) have considered this effect and shown its modification to the small-amplitude dust-acoustic solitons (DASs), as well as large-amplitude dust-acoustic solitary waves (DASW) by either the Sagdeev pseudopotential method (Sagdeev (1966)) or the reductive perturbation method (Washimi and Taniuti (1966)). Chen and Lai (2007) discussed the role of plasma in formation of large scale structures caused in the Universe by assuming that the latter is composed of electrons and
baryons/ions only. Such an assumption is inadequate since a fully ionized universe contains electrons, positrons, ions and micron sized charged grains (Miller and Williams (1993); Evans (1994)). Therefore, electrons, ions, positrons and dust grains forming four component plasmas are believed to exist in active galactic nuclei, pulsar magnetospheres, interstellar clouds, supernova environments as well as in laboratory experiments of cluster explosions by intense laser beams (Weinberg (1972); Shukla et al. (1997); Tajima and Shibata (1997); Cho et al. (2000); Shukla and Marklund (2004); Higdon et al. (2009)).

Linear and nonlinear propagation of waves in dusty plasmas in the presence of positrons has been studied by many authors (Shukla et al. (1997); Cho et al. (2000); Mirza and Khan (2002); Shukla and Marklund (2004); Shin and Jung (2006); Ghosh and Bharuthram (2008)). Appreciable amounts of dust in a plasma system exhibit collective behavior and give rise to new low frequency modes. The large dust mass by comparison with ions and electrons/positrons introduces new scale lengths in the system which are non-existent in conventional electron-ion plasma as well as in e-p-i plasma. In dusty plasmas, it is well known that the two normal modes of unmagnetized, weakly coupled plasmas are the dust-acoustic (DA) and dust ion acoustic (DIA) waves. These modes were theoretically predicted by Rao et al. (1990), as well as by Shukla and Silin (1992) and were confirmed by a number of experimentalists using different plasma devices (Barkan et al. (1995); Pieper and Goree (1996); Praburam and Goree (1996); Prabhakara and Tanna (1996); Thompson et al. (1997); Merlino et al. (1998)). In the DA wave, the inertialess electrons/positrons and ions provide the restoring force while the dust mass gives the inertia. The frequency range of this wave is much lower and the phase velocity much smaller than the ion acoustic velocity. Among the most
studied low frequency modes, dust-acoustic solitons and dust-acoustic solitary waves have been the focus of attraction in the recent past. Solitons owe their existence to the delicate balance of wave dispersion by nonlinearity. There has been a prolific literature on dust-acoustic solitary waves (DASWs) and on other nonlinear wave structures in last decade (Mamun et al. (1996a); El-Labany and El-Taibany (2003a); El-Labany and El-Taibany (2003b); El-Labany et al. (2003b); El-Labany and El-Taibany (2004); El-Labany et al. (2004b); Akhtar et al. (2007); Gill et al. (2007); Verheest and Pillay (2008); Zhang et al. (2008)). These DASWs have also been investigated experimentally (Barkan et al. (1995); Pieper and Goree (1996); Prabhakara and Tanna (1996); Merlino et al. (1998); Nakamura et al. (2001)). Theoretically the best known paradigms invoked to investigate nonlinear wave behavior are different versions of nonlinear partial differential equations, some of which are integrable. Particularly for solitons, we adopt Korteweg-de-Vries (KdV), modified Korteweg-de-Vries (mKdV) and nonlinear Schrödinger equations (NLSE). Reductive perturbation technique is used to derive such equations. The KdV equation is derived in small amplitude limit and admits both ion-acoustic or dust-acoustic solitary wave modes (DASWs) depending on the conditions. The pseudopotential method is used for the large amplitude solitons. In this method, full nonlinearity is taken into consideration. There are some excellent review articles on solitons that have appeared in literature (Mamun and Shukla (2002d); Shukla (2003); Shukla and Mamun (2003)).

It is believed that double layers occur naturally in a space plasma environments (e.g. solar wind, extra galactic jets, auroras etc.). Double layers may accelerate, decelerate or reflect the plasma particles. Due to Double layers relevance in cosmic plasmas and space properties of plasma, the formation of double layers has been
9.2 Fluid Model

received a great deal of interest (Temerin et al. (1982); Kim (1987); Raadu (1989); Mishra et al. (2002); Cattaert et al. (2005); Amour and Tribeche (2009)). It has been argued that the large portion of the total potential on auroral field lines explained by small amplitude double layers may provide the explanation about the fine structure of auroral kilometric radiation.

Since nonthermal distribution of particles (Cairns et al. (1995a); Cairns et al. (1995b); Volosevich et al. (2002)) is a common feature of space and astrophysical environments, therefore it is important to investigate the characteristics of DASWs with such distributions. The aim of the present investigation is to study the role of nonthermal ion distribution and positron density on the solitary wave behavior as well as double layers in dusty plasma considering dust charge as variable. In the research work reported in this part of the thesis, the characteristics of dust-acoustic solitary waves (DASWs) and double layers are studied in an unmagnetized plasma consisting of electrons, positrons, ions and dust species. This is done by deriving the potential using Sagdeev pseudopotential method. Ions are treated as nonthermal and variable dust charge is considered. The Korteweg de-Vries (KdV) equation and Double Layer expression is obtained by carrying out series expansion of potential. Both DASWs and DLs are sensitive to variation of nonthermal parameter and positron density. The present investigation may have relevance in both laboratory and astrophysical plasmas.

9.2 Fluid Model

The dusty plasma, we are studying, consists of four components, extremely massive-highly negatively charged dust grains, maxwellian electrons and positrons and nonthermal ions.
Chapter 9. Dust-acoustic solitary waves in a dusty plasma in the presence of positrons

The fluid model is described by (continuity, momentum and Poisson’s equations)

\[
\frac{\partial \tilde{n}_d}{\partial t} + \frac{\partial (\tilde{n}_d \tilde{u}_d)}{\partial x} = 0 \tag{9.1}
\]

\[
\frac{\partial \tilde{u}_d}{\partial t} + \tilde{u}_d \frac{\partial \tilde{u}_d}{\partial x} = \frac{e \tilde{Z}_d}{m_d} \frac{\partial \tilde{\phi}}{\partial x} \tag{9.2}
\]

\[
\frac{\partial^2 \tilde{\phi}}{\partial x^2} = 4\pi e (\tilde{Z}_d \tilde{n}_d + \tilde{n}_e - \tilde{n}_i - \tilde{n}_p) \tag{9.3}
\]

Charge neutrality at equilibrium yields

\[n_{i0} + n_{p0} = n_{e0} + Z_{d0} n_{d0} \tag{9.4}\]

where \(n_{i0}, n_{p0}, n_{e0}\) and \(n_{d0}\) are the unperturbed ions, positrons, electrons and dust number densities respectively and \(Z_{d0}\) is the unperturbed number of charges on the dust grain, measuring in units of the electron charge. We can also write charge neutrality condition as

\[
\frac{n_{i0}}{Z_{d0} n_{d0}} + \frac{n_{p0}}{Z_{d0} n_{d0}} = \frac{n_{e0}}{Z_{d0} n_{d0}} + 1 \tag{9.5}
\]

\[\delta_i + \delta_p = \delta_e + 1 \tag{9.6}\]

where \(\delta_e = \frac{n_{e0}}{Z_{d0} n_{d0}}, \delta_p = \frac{n_{p0}}{Z_{d0} n_{d0}}\) and \(\delta_i = \frac{n_{i0}}{Z_{d0} n_{d0}}.\)

The normalized densities of electrons, positrons and ions are

\[n_e = \delta_e e^{\exp(\beta_1 \phi)} \tag{9.7}\]

\[n_p = \delta_p e^{\exp(-\beta_2 \phi)} \tag{9.8}\]

\[n_i = \delta_i (1 + \beta \phi + \beta \phi^2) e^{\exp(-\phi)} \tag{9.9}\]

where \(\beta_1 = \frac{T_i}{T_e}, \beta_2 = \frac{T_i}{T_p}\) and \(\beta = \frac{4\gamma}{1 + 3\gamma}\). Parameter \(\beta\) represents the nonthermality of ion distribution.
9.3 Study of dust charge variation

The scaling parameters are $u_d = \overline{u_d}$, $n_d = \overline{n_d}$, $Z_d = \overline{Z_d}$, $x = \overline{x_d}$, and $t = \overline{t}$. Debye length $\lambda_D = \sqrt{\left(T_i/4\pi Z_d n_d e^2\right)}$ and inverse of plasma frequency, $\omega_{pd}^{-1} = \sqrt{\left(m_d/4\pi n_d Z_d^2 e^2\right)}$ respectively. Here $c_d = \sqrt{(T_i Z_d / m_d)}$. The electrostatic potential $\tilde{\phi}$ is normalized by $T_i/e$ and $\gamma$ is the nonthermal parameter. To investigate this problem, we have considered nonthermal ions (Cairns et al. (1995a;1995b)).

The normalized equations for the given fluid model are as follows:

\[ \frac{\partial n_d}{\partial t} + \frac{\partial (n_d u_d)}{\partial x} = 0 \]  \hspace{1cm} (9.10)

\[ \frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} = Z_d \frac{\partial \tilde{\phi}}{\partial x} \]  \hspace{1cm} (9.11)

\[ \frac{\partial^2 \tilde{\phi}}{\partial x^2} = Z_d n_d + n_e - n_i - n_p \]  \hspace{1cm} (9.12)

9.3 Study of dust charge variation

The charge current balance equation for the variable dust charge $Q_d$ (Melandso et al. (1993)) is

\[ \left( \frac{\partial}{\partial t} + u_d \frac{\partial}{\partial x} \right) Q_d = I_e + I_i + I_p \]  \hspace{1cm} (9.13)

The charging currents are originated from the electrons, positrons and ions, reaching the grain surface. The dust charging time is of the order of $10^{-9}$ s, while dust motion time is of the order of $10^{-3}$ s. Thus, the dust charge can quickly reach the local equilibrium at which the currents from the electrons, positrons and ions to the dust grains are balanced. The current balance equation implies

\[ I_e + I_i + I_p \approx 0 \]  \hspace{1cm} (9.14)

We further suppose that the streaming velocities of electrons, positrons and ions are much smaller than the thermal velocities. The electron, ion and positron
currents are

\[
I_e = -e \pi r^2 \left( \frac{8 T_e}{\pi m_e} \right)^{1/2} n_e \exp \left( \frac{e \Phi}{T_e} \right)
\]

\[
I_i = e \pi r^2 \left( \frac{8 T_i}{\pi m_i} \right)^{1/2} n_i \left( 1 - \frac{e \Phi}{T_i} \right)
\]

\[
I_p = e \pi r^2 \left( \frac{8 T_p}{\pi m_p} \right)^{1/2} n_p \exp \left( -\frac{e \Phi}{T_p} \right)
\]

(9.15)

where \( \Phi \) denotes the dust grain surface potential relative to the plasma potential \( \phi \). From the current balance equation and following Xie et al. (1999), we have

\[
\tilde{\alpha}_1 \delta (1 + \beta \phi + \beta^2 \phi^2) \exp(-\phi)(1-\psi) - \exp(\beta_1 (\phi+\psi)) + \tilde{\alpha}_2 \delta_1 \exp(-\beta_2 (\phi+\psi)) = 0 \quad (9.16)
\]

where \( \psi = e \Phi / T_i \), \( \tilde{\alpha}_1 = \sqrt{\beta_1 / \mu_i} \), \( \delta = n_{i0} / n_{e0} \), \( \delta_1 = n_{p0} / n_{e0} \), and with the mass ratio of ion to electron \( \mu_i = m_i / m_e \simeq 1843 \). \( \tilde{\alpha}_2 = \sqrt{\beta_1 \mu_p / \beta_2} \), \( \mu_p = m_e / m_p \), \( \beta_2 = T_i / T_p \).

By using equation (9.16), we can calculate the dust charge \( Q_d = C \Phi \), where \( C \) is capacitance of dust grain (\( C = \pi r \)) and

\[
Z_d = \frac{\psi}{\psi_0} \quad (9.17)
\]

where \( \psi = \psi_0 \) at \( \phi = 0 \) is the dust surface floating potential with respect to the unperturbed plasma potential at an infinite distance from the dust grain surface. In equation (9.16), substituting \( \phi = 0 \) to determine the value of \( \psi_0 \), we have

\[
\tilde{\alpha}_1 \delta (1 - \psi_0) - \exp(\beta_1 \psi_0) + \tilde{\alpha}_2 \delta_1 \exp(-\beta_2 \psi_0) = 0 \quad (9.18)
\]

Now expanding \( \psi \) near \( \psi_0 \), \( Z_{d1}, Z_{d2} \) and \( Z_{d3} \) from equation (9.17) can be obtained as follows:

\[
Z_{d1} = \Upsilon_1 \phi_1, \quad \Upsilon_1 = \frac{\Upsilon_b}{\Upsilon_a \psi_0}, \quad (9.19)
\]

where

\[
\Upsilon_a = [\tilde{\alpha}_1 \delta + \beta_1 \exp(\beta_1 \psi_0) + \tilde{\alpha}_2 \delta_1 \beta_2 \exp(-\beta_2 \psi_0)]
\]
and

\[ \Upsilon_b = -[\beta_1 \exp(\beta_1 \psi_0) + \alpha_2 \delta_1 \beta_2 \exp(-\beta_2 \psi_0) - \alpha_1 \delta(1 - \psi_0)(\beta - 1)]. \]

\[ Z_{d2} = \Upsilon_1 \phi_2 + \Upsilon_2 \phi_1^2, \quad \Upsilon_2 = \frac{\Upsilon_c}{\Upsilon_a \psi_0}, \quad (9.20) \]

with

\[ \Upsilon_c = \Upsilon_{c1} + \Upsilon_{c2} + \Upsilon_{c3}, \]

\( \Upsilon_{c1}, \Upsilon_{c2}, \Upsilon_{c3} \) are given in Appendix-F.

\( \psi_0 \) is always negative for the negatively charged dust grains, so the condition

\[ 1 \geq \alpha_1 \delta \quad (9.21) \]

must be satisfied.

\[ Z_{d3} = \Upsilon_1 \phi_3 + 2\Upsilon_2 \phi_1 \phi_2 + \Upsilon_3 \phi_1^3, \quad \Upsilon_3 = \frac{\Upsilon_d}{\Upsilon_a \psi_0}, \quad (9.22) \]

with

\[ \Upsilon_d = \Upsilon_{d1} + \Upsilon_{d2} + \Upsilon_{d3} + \Upsilon_{d4} + \Upsilon_{d5} + \Upsilon_{d6} \]

where \( \Upsilon_{d1}, \Upsilon_{d2}, \Upsilon_{d3}, \Upsilon_{d4}, \Upsilon_{d5} \) and \( \Upsilon_{d6} \) are given in Appendix-F. The self-consistent dust charge variation with the plasma potential would have a remarkable effect on the small amplitude as well as on the large amplitude solitary waves. Figure 9.1 depicts the dependence of normalized dust charge number \( Z_d(= \frac{\phi}{\psi_0}) \) on the plasma potential disturbance \( \phi \) for \( \beta = 0.1, \beta_1 = 0.001, \beta_2 = 0.1, \delta_1 = 0.1 \) and \( \delta = 0.3 \). For negative plasma potential disturbance, as its strength increases, \( Z_d \) would decrease from one to zero. The cut off occurs at \( \phi \approx -4 \). However, for the positive plasma potential with increasing disturbance strength, first \( Z_d \) would increase quickly with a larger slope, then gradually slow down with a smaller
Figure 9.1: Variation of normalized dust charge number $Z_d$ with plasma potential disturbance $\phi$ for fixed $\beta = 0.1$, $\beta_1 = 0.001$, $\beta_2 = 0.1$, $\delta_1 = 0.1$ and $\delta = 0.3$.

slope. From figure 9.1, it is clear that the dust charge is very sensitive to the small disturbance of $\phi$ around the unperturbed state $\phi = 0$, but gradually tends to saturate for the large plasma potential. This clearly explains how the variable dust charge influences the shape of solitons and solitary waves.

9.4 Sagdeev pseudopotential Method

In order to investigate the properties of large amplitude dust-acoustic solitary waves, we assume that all fluid variables in evolution equations (9.10), (9.11) and (9.12) depend only on a single variable $\xi = x - Mt$ (where $M$ is the Mach number, i.e., the pulse propagation velocity of the solitary wave normalized to the dust-
acoustic speed \( c_d \). The equation of continuity transforms as

\[
-M \frac{\partial n_d}{\partial \xi} + \frac{\partial (n_d u_d)}{\partial \xi} = 0
\]  

(9.23)

On further simplification above equation, yields

\[
n_d = \frac{M}{M - u_d}
\]  

(9.24)

Similarly, the equation of motion for dust transforms as

\[
-M \frac{\partial u_d}{\partial \xi} + u_d \frac{\partial u_d}{\partial \xi} = Z_d \frac{\partial \phi}{\partial \xi}
\]  

(9.25)

which when integrated, yields

\[
V_d(\phi) = \frac{(M - u_d)^2}{2} - \frac{M^2}{2}
\]  

(9.26)

To obtain the expressions for dust density \( n_d \) and \( V_d(\phi) \), we have used the appropriate boundary conditions, namely, \( \phi \to 0 \), \( n_d \to 1 \) and \( u_d \to 0 \) as \( \xi \to \pm \infty \). Furthermore,

\[
V_d(\phi) = \int_0^\phi Z_d d\phi = \frac{1}{\psi_0} \int_0^\phi \psi(\phi) d\phi
\]  

(9.27)

From equations (9.24) and (9.26), we obtain the dust density as

\[
n_d = \left(1 + \frac{2V_d(\phi)}{M^2}\right)^{-1/2}
\]  

(9.28)

using equation (9.28) in Poisson’s equation (9.12), we get

\[
\frac{\partial^2 \phi}{\partial \xi^2} = Z_d \left(1 + \frac{2V_d(\phi)}{M^2}\right)^{-1/2} + n_e - n_i - n_p
\]  

(9.29)

Substituting equations (9.7-9.9) into equation (9.29), multiplying the resulting equation by \( d\phi/d\xi \), integrating and applying the boundary conditions, \( d\phi/d\xi \to 0 \) at \( \xi \to \pm \infty \), we get

\[
\frac{1}{2} \left(\frac{d\phi}{d\xi}\right)^2 + V(\phi, M) = 0,
\]  

(9.30)
where the pseudopotential \( V(\phi, M) \) is

\[
V(\phi, M) = M^2 \left[ 1 - \left( 1 + \frac{2V_d(\phi)}{M^2} \right)^{1/2} \right] + \frac{1}{\beta_1(\delta_1 + \delta - 1)} [1 - \exp(\beta_1 \phi)]
\]

\[
+ \frac{\delta_1}{\beta_2(\delta_1 + \delta - 1)} [1 - \exp(-\beta_2 \phi)]
\]

\[
+ \frac{\delta}{(\delta_1 + \delta - 1)} (1 - \exp(-\phi)) (1 + 3\beta)
\]

\[
- \frac{\delta}{\delta_1 + \delta - 1} (3\beta \phi + \beta \phi^2) \exp(-\phi),
\]

Equation (9.31) can be regarded as an energy-balance equation for an oscillating particle of unit mass with position \( \phi \), time \( \xi \), velocity \( \frac{d\phi}{d\xi} \) and \( V(\phi, M) \) given by equation (9.31). Conditions for the existence of a localized soliton solution of (9.31) require:

(i) \( V(\phi, M) = 0, \frac{dV(\phi, M)}{d\phi} = 0 \) and \( \frac{d^2V(\phi, M)}{d\phi^2} < 0 \) at \( \phi = 0 \), i.e. the fixed point at the origin is unstable;

(ii) that there exists a non-zero \( \phi_m \), the maximum (or minimum) value of \( \phi \) at which \( V(\phi_m, M) = 0 \); and

(iii) \( V(\phi, M) < 0 \) when \( \phi \) lies between zero and \( \phi_m \).

(iv) for double layers, both \( V(\phi_m, M) \) and \( \frac{dV(\phi_m, M)}{d\phi} \) must be zero.

The requirement in (i) leads to the “soliton condition”, \( M > M_s \), where

\[
M_s = \left( \frac{\delta_1 + \delta - 1}{\beta_1 + \beta_2 \delta_1 + \delta(1 - \beta)} \right)^{1/2}
\]

is the lower limit of Mach number below which no solitons (or double layers) can exist. We have analysed numerically the variation of the Mach number (lower velocity limit) \( M_s \) with positron density parameter \( \delta_1 \) and parameter \( \beta \) for other fixed values.

The variation of \( M_s \) with \( \delta_1 \) (positron density parameter) is shown in figure 9.2 for two different values of \( \beta_2 \). It is seen that the critical value of the Mach number
9.4. Sagdeev pseudopotential Method

Figure 9.2: Variation of the critical Mach number $M_s$ with $\delta_1 (= n_{p0}/n_{e0})$ for fixed $\beta = 0.1$, $\beta_1 = 0.001$, and $\delta = 0.3$.

Figure 9.3: Variation of the critical Mach number $M_s$ with $\beta$ for fixed $\beta_1 = 0.001$, $\beta_2 = 0.1$, and $\delta = 0.3$. 
decreases with increase in $\delta_1$ and $\beta_2$. However, it is observed that the value of $M_s$ increases with $\beta$ for a given value of $\delta_1$ but decreases with increase in $\delta_1$ (see figure 9.3). There are five terms for the expression $V(\phi, M)$ in (9.31). The first term in $V(\phi, M)$ enclosed in parentheses owe their physical origin to the dust charge. The electron contribution is through the second term while fourth and fifth terms are respectively the contribution of the ions. The positron contribution is in the third term. The dust charge enter the expression for $V(\phi, M)$ through $V_d(\phi)$. Depending on the chosen set of parameters, it is the relative contributions of these terms that lead to the formation of different types of soliton we obtain. However, it is always possible to vary parameters leading to the dominance of a particular term over the others. We have chosen the parameters $\beta_1 = 0.001$, $\delta = 0.3$, and $M = 2$ for numerical computation.

Figure 9.4 shows the variation of $V(\phi, M)$ as a function of $\phi$ for different values of $\delta_1$. Compressive and rarefactive solitons coexist for the chosen set of parameters. With increase in the positron density the amplitude of both positive and negative solitons increases. Decrease in $\delta_1$ leads to only one type of soliton. The contribution of last two terms in $V(\phi, M)$ is insignificant for small values of $\phi$. However, as the value of $\phi$ approaches midway through $\phi_m$, the contribution of these terms becomes comparable to that of the dust charge terms. Once $\phi$ approaches $\phi_m$, the ion dynamics become comparably important. This leads to a monotonic increase in $V(\phi, M)$, with the result that for $\phi > \phi_m$, $V(\phi, M)$ becomes positive. We have also studied the variation of $V(\phi, M)$ as a function of $\phi$ for different values of $M$. From figure 9.5, it is observed that there is a coexistence of compressive and rarefactive solitons, however with increase in the Mach number leads to disappearance of firstly rarefactive solitons afterwards compressive solitons. Beyond $M = 2.75$
9.4. Sagdeev pseudopotential Method

Figure 9.4: Variation of pseudopotential $V(\phi, M)$ with $\phi$ for different values of $\delta_1 (= n_{p0}/n_{e0})$ and fixed $M = 2$, $\beta = 0.4$, $\delta = 0.3$, $\beta_2 = 0.1$ and $\beta_1 = 0.001$.

neither compressive nor rarefactive solitons exist, this limits the upper limit of Mach number.

In figures 9.6 (a) and (b), we have depicted the variation of the maximum electrostatic (ES) potential $\phi_m$ of negative and positive potential solitary waves versus $\delta_1$. For negative potential solitary waves, maximum potential $\phi_m$ decreases with increase in the Mach number value but increases with increase in the value of $\delta_1$. On the other hand, for positive potential solitary waves maximum potential $\phi_m$ increases monotonically with increase in the Mach number value and increase in the value of $\delta_1$. The variation of maximum potential in these figures confirms the above mentioned discussion for the figures 9.4 and 9.5.
Figure 9.5: Plot of pseudopotential $V(\phi, M)$ with $\phi$ for different values of $M$ and with $\beta = 0.4$, $\delta_1 = 0.1$, $\delta = 0.3$, $\beta_2 = 0.1$ and $\beta_1 = 0.001$.

9.5 Small amplitude dust-acoustic solitons and double layers

To study small amplitude solitary wave structures, we carry out a series expansion of $V(\phi, M)$ about the origin ($\phi = 0$). To fourth order this leads to

$$
\frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 + \tilde{A}\phi^2 + \tilde{B}\phi^3 + \tilde{C}\phi^4 = 0,
$$

(9.33)

where

\[
\tilde{A} = -\frac{\gamma_1}{2} + \frac{1}{2M^2} - \frac{(\beta_1 + \delta_1\beta_2)}{2(\delta_1 + \delta - 1)} - \frac{\delta(1 - \beta)}{2(\delta_1 + \delta - 1)}
\]

(9.34)

\[
\tilde{B} = \frac{\gamma_1}{2M^2} - \frac{\gamma_2}{3} - \frac{1}{2M^4} - \frac{(\beta_1^2 - \beta_2^2)}{6(\delta_1 + \delta - 1)} + \frac{\delta}{6(\delta_1 + \delta - 1)}
\]

(9.35)

and

\[
\tilde{C} = -\frac{\gamma_3}{4} + \frac{\gamma_1^2}{8M^2} + \frac{\gamma_2}{3M^2} - \frac{\gamma_1}{4M^4} + \frac{5}{8M^6} - \frac{\beta_1^3 + \beta_2^3}{24(\delta_1 + \delta - 1)} - \frac{\delta}{24(\delta_1 + \delta - 1)}(1 + 3\beta)
\]

(9.36)
9.5. Small amplitude dust-acoustic solitons and double layers

Figure 9.6: (a) Variation of maximum potential $\phi_m$ for negative potential solitary waves and (b) for positive potential solitary waves with $\delta_i$ for different values of the Mach number $M$ for fixed $\beta = 0.4$, $\delta = 0.3$, $\beta_2 = 0.1$ and $\beta_1 = 0.001$. 
For small amplitude solitons, we first assume that the fourth term in equation (9.33) is small enough to be neglected and only consider

\[ \frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 + \tilde{A}\phi^2 + \tilde{B}\phi^3 = 0, \]

(9.37)

which leads to the usual Korteweg-de Vries (KdV)-type solution

\[ \phi(\xi) = -\frac{\tilde{A}}{\tilde{B}} \text{sech}^2\left((-\tilde{A}/2)^{1/2}\xi\right) \]

(9.38)

Here, we require \( \tilde{A} < 0 \) for real solutions in equation (9.38), and \( \tilde{B} \) nonzero, since we require \( \phi \to 0 \) as \( \xi \to \pm\infty \). This approach, which is valid for weak (small amplitude) solitons, is similar to the reductive perturbation technique that results in the Korteweg-de Vries equation of the form of equation (9.37). The maximum soliton potential \( \phi_{Im} \) and width are given by \( |\tilde{A}| / |\tilde{B}| \) and \( \sqrt{-\tilde{A}/\tilde{B}} \), respectively, with the sign of the potential being dependent on the sign of \( \tilde{B} \). In particular, for \( \tilde{B} > 0 \) it follows that \( \phi \) is positive, while for a negative potential soliton we require \( \tilde{B} < 0 \). In other words, the sign of the coefficient of \( \phi^3 \) in the Taylor expansion of \( V(\phi, M) \) about \( \phi = 0 \) determines the sign of the potential of the small amplitude solitons that exist in the plasma model. In figure 9.7, we have plotted the soliton potential \( \phi_{Im} \) versus \( \delta_1 \) for different mach numbers. Amplitude of KdV soliton decreases with increase in \( \delta_1 \) for a given Mach number, and for a given value of \( \delta_1 \), amplitude decreases with increase in Mach number values. Plots of soliton potential \( \phi_{Im} \) versus \( \delta_1 \) for different \( \beta \) is observed in figure 9.8. The soliton potential \( \phi_{Im} \) decreases with \( \delta_1 \) for finite \( \beta \).

Let us now consider the case of possible small amplitude double layers. It is well known that double layers may act as limits of a sequence of solitons, and can thus give rise to a limit of an existence domain for solitons. Assuming that the double layer condition \( V(\phi_m, M) = V'(\phi_m, M) = 0 \) (existence of a second double
small amplitude dust-acoustic solitons and double layers

Figure 9.7: Variation of soliton potential ($\phi_{Im}$) in KdV with $\delta_1 (= n_{p0}/n_{e0})$ for different values of $M$ and with $\beta = 0.4$, $\beta_2 = 0.1$, $\delta = 0.3$ and $\beta_1 = 0.001$.

Figure 9.8: Variation of soliton potential ($\phi_{Im}$) in KdV with $\delta_1 (= n_{p0}/n_{e0})$ for different values of $\beta$ and with $M = 2$, $\beta_2 = 0.1$, $\delta = 0.3$ and $\beta_1 = 0.001$. 
root at $\phi_m$) applies to equation (9.33) at $\phi = \phi_m \neq 0$, one obtains $\phi_m^2 = \tilde{A}/\tilde{C}$ and $\phi_m = -\frac{\tilde{B}}{2\tilde{C}}$, or simply $B^2 = 4\tilde{A}\tilde{C}$. Using this transformation, equation (9.33) can then be written in the form

$$\frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 + \tilde{C}\phi^2(\phi - \phi_m)^2 = 0,$$

which has a solution

$$\phi = \phi_{2m} \left[ 1 - \tanh \left( (-\tilde{A}/2)^{1/2}\xi \right) \right]$$

provided $\tilde{A} < 0$. Also, for real values of $\phi_m$, $\tilde{C}$ must be negative, and therefore the sign of the double layer given by equation (9.40) depends solely on whether $\tilde{B}$ is negative or positive. Here $\phi_{2m} = -\frac{\tilde{B}}{4\tilde{C}}$. In order to study the double layer solution, we have studied the variation of double layer amplitude ($\phi_{2m}$) with $\delta_1$ for different Mach numbers (see figure 9.9). The amplitude of double layers
9.6 Conclusions

We have provided a thorough analysis from first principles of the occurrence of large amplitude dust-acoustic solitary waves and double layers in an unmagnetized plasma consisting of electrons, nonthermal ions, positrons and dust grains with variable charge. A Sagdeev pseudopotential approach was used to derive an energy-balance like equation. The range of allowed values of the Mach number, wherein solitary waves may exist, was determined. We have traced the effect of positron concentration and solitary speed with variable dust charge on the characteristics increases with increase in $\delta_1$ for finite $M$. Similarly we have plotted the double layer amplitude ($\phi_{2m}$) with $\delta_1$ for different values of nonthermal parameters, i.e., $\beta$ (see figure 9.10). The amplitude of double layers increases with increase in $\delta_1$ for finite $\beta$ and decreases with increase in $\beta$ for given $\delta_1$.

Figure 9.10: Variation of double layer amplitude ($\phi_{2m}$) in DL solution with $\delta_1 (= n_{p0}/n_{e0})$ for different values of $\beta$ and with $M = 2$, $\beta_2 = 0.1$, $\delta = 0.3$ and $\beta_1 = 0.001$. 

9.6 Conclusions
of dust-acoustic solitary structures. Our results are summarized as follows:

(i) Both large positive and negative potential solitary structures may co-exist.

(ii) For fixed values of nonthermal parameter $\beta$ and other physical parameters, positive potential soliton amplitude increases with increase in the positron concentration and amplitude of negative potential solitons increases on negative side of $\phi$ axis (i.e., decreases with increase in positron concentration).

(iii) The Korteweg de-Vries (KdV) equation and Double layer expression are derived. Numerical results reveal that the amplitude and width of double layers are significantly affected by positron concentration (via $\delta_1$) and nonthermal parameter $\beta$.

(iv) Amplitude of double layers decreases monotonically with solitary speed (Mach number) and increases with increase in positron concentration.

This study may shed some light on the properties of dust-acoustic solitary waves in space and astrophysical plasmas, where the combined presence of electrons, nonthermal ions, positrons and charged dust grains may be encountered.