Chapter 2

Localized coherent nonlinear wave structures in dusty plasma with nonthermal ions

2.1 Introduction

The past decade has witnessed a rapid growth in the field of dusty plasmas. In dusty plasma, a third charged species with size ranging from nanometers to several hundred microns in diameter is present. The history, occurrence and characterization of dusty plasma in space and laboratory environment are well described and documented in the recent publications (Verheest (2000); Shukla and Mamun (2002)). Collective processes, like low frequency modes in dusty plasma, have received a great deal of attention during the last many years. The presence of charged dust particles, introduces many interesting features to collective processes and modify the propagation of waves, instabilities, etc. in dusty plasma. Many low frequency modes have been investigated, particularly, dust ion-acoustic (DIA) waves (Shukla and Silin (1992); Shukla (1992)), dust-acoustic (DA) waves (Rao et al. (1990)), dust-coulomb (DC) waves (Rao (1999)), dust-lattice waves (Melandso et al. (1993)) etc. These are the normal modes of unmagnetized dusty plasma.
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and have been described in detail (Verheest (2000); Shukla and Mamun (2002)). In most of the normal modes studied, variation of charge has not been taken into account. Varma et al. (1993) were the first to study electrostatic oscillations in the presence of grain-charge perturbations in dusty plasmas.

The linear theory of waves, based on small amplitude approximation breaks down completely when amplitude becomes large. At this stage, non-linear effects play an important role, leading to the formation of coherent nonlinear wave structures. Presence of nonlinearity, dispersion and dissipation when adequately balanced manifest themselves in the formation of coherent nonlinear wave structures like solitons, shock waves, double layers, vortices etc. Most widely investigated of these structures are dust-acoustic solitons (DASs) and dust-acoustic solitary waves (DASWs), for which methodology, and physics issues are well presented in a literature (Shukla and Mamun (2002)). Some form of reductive perturbation technique is used to derive Korteweg-de-Vries (KdV) or nonlinear Schrödinger equation (NLSE). In most of the investigations, KdV model has been used to study DASs and DASWs. However, Amin et al. (1998b) used reductive perturbation technique to derive NLSE and studied the modulational instability (MI) of DASWs and DIA waves. For the sake of simplicity, dust was treated cold with constant charge on it. Introduction of finite dust temperature has been considered in the recent investigations (Roychoudhury and Mukherjee (1997); Gill and Kaur (2000)). However, it has been pointed out that dust charge should be considered as a dynamical variable due to the variety of mechanisms (Nejoh (1997b); Xie et al. (1999)). This will result in modification of collective properties of a dusty plasma. Ignoring the dust temperature, Xie et al. (1999) have studied DASWs and double layers in dusty plasma with variable dust charge and two temperature
ions with Boltzmann distributions. However, it has been found that electron and ion distributions play crucial role in the formation of nonlinear structures. Several investigations have recently been reported in plasmas with non-Maxwellian particle distributions (Mamun et al. (1996a); Mamun et al. (1996b); Mamun and Cairns (1996); Mamun (1997); Mamun (1998a); Mamun (1998b); Mamun (2000); Mamun and Shukla (2002a)). Gill et al. (2004) have studied the properties of ion-acoustic solitons and double layers in a plasma consisting of warm positive and negative ions with different concentration of masses, charged states and nonthermal electrons.

Among the nonlinear structures, double layers (DLs) are of considerable importance because of their relevance to cosmic applications (Alfven and Carlqvist (1967); Tenerin et al. (1982); Borovsky (1984); Carlqvist (1986); Mishra et al. (2002)). In space plasmas, they are considered as a source of earth’s aurora as well as responsible for auroral electron precipitation. Furthermore, DLs provide a mechanism for acceleration of charged particle. Among the other nonlinear structures like solitons, DLs are least studied (Goswami and Bujarbarua (1985); Bharuthram and Shukla (1986); Jain et al. (1990); Yadav and Sharma (1991)). Recently, DLs and DASs in dusty plasmas have attracted a great deal of attentions (Roychoudhury and Chatterjee (1999); Xie et al. (1999); Shukla (2000); Shukla and Mamun (2001); El-Labany and El-Taibany (2003a); El-Labany et al. (2004b); Gill et al. (2004)). There have been reviews on these nonlinear wave structures reported recently (Shukla and Mamun (2003); Shukla (2003)).

Numerical simulations have established beyond doubt that particles may not follow Maxwellian distribution. As mentioned above the electron and the ion distributions play a crucial role in characterizing the physics of the wave structures. They offer considerable increase in richness and variety of wave motion which can
exist in plasma and further significantly influence the conditions required for the formation of solitons and double layers. Clearly there is a departure from Boltzmann distributions. Several investigations have explored the role played by such non-Maxwellian distribution on solitons in dusty as well as multi species plasmas. With the observation of solitary wave structures with density depression, the role of nonthermal electron distribution on characterization of solitary wave/solitons, were reported (Cairns et al. (1995a); Cairns et al. (1995b); Volosevich et al. (2002)).

Nonthermal distributions are common features of the auroral zone (Lundin et al. (1987); Hall et al. (1991)). Mechanism for the formation of nonthermal particle distribution in space plasmas is still a central problem. The aim of the present investigation is to study the role of nonthermal ion distribution on the solitary wave behavior as well as double layers in dusty plasma considering dust charge as variable.

In this chapter, the characteristics of dust-acoustic solitary waves (DASWs) and double layers are studied. Ions are treated as nonthermal and variable dust charge is considered. The Korteweg de-Vries (KdV) equation is derived using reductive perturbation method. It is noticed that compressive solitons are obtained upto certain range of relative density $\delta (= n_i/n_e)$ beyond which rarefactive solitons are observed. The study is further extended to investigate the possibility of double layers (DLs). Only compressive double layers are permissible. Both DASWs and DLs are sensitive to variation of nonthermal parameter.

### 2.2 Governing Equations

In the given plasma model, dusty plasma with extremely massive, high negatively charged dust grains, electrons and nonthermal ions are considered. The unnormal-
ized dusty plasma model is described as follows:

\[
\frac{\partial n_d}{\partial t} + \frac{\partial(n_d u_d)}{\partial x} = 0
\]

\[
\frac{\partial n_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} = \frac{eZ_d}{m_d} \frac{\partial \phi}{\partial x}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = 4\pi e(Z_d n_d + n_e - n_i)
\]

The charge conservation implies the following equilibrium equation:

\[
n_{i0} = Z_{d0} n_{d0} + n_{e0}
\] (2.1)

where \(n_{i0}, n_{e0}\) and \(n_{d0}\) are the unperturbed ions, electrons and dust number densities respectively and \(Z_{d0}\) is the unperturbed number of charges on the dust grain.

Followings are the normalized continuity, momentum and Poisson’s equations governing the dynamics of dusty plasma:

\[
\frac{\partial n_d}{\partial t} + \frac{\partial(n_d u_d)}{\partial x} = 0
\] (2.2)

\[
\frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} = Z_d \frac{\partial \phi}{\partial x}
\] (2.3)

\[
\frac{\partial^2 \phi}{\partial x^2} = Z_d n_d + n_e - n_i
\] (2.4)

where \(n_d, u_d, \phi\) and \(Z_d\) are the dust number density, dust particle velocity, electrostatic potential and the variable charge number of dust grains, respectively.

For the sake of convenience, we here use the procedure of Xie et al. (1999). The electrons and ions obey the Boltzmann distribution and nonthermal distribution, respectively, as follows:

\[
n_e = \frac{n_{e0}}{Z_{d0} n_{d0}} exp(\beta_1 \phi)
\] (2.5)

\[
n_i = \frac{n_{i0}}{Z_{d0} n_{d0}} (1 + \beta \phi + \beta \phi^2) exp(-\phi).
\] (2.6)
Parameters $\beta_1$ and $\beta$ are defined, respectively, as follows:

\[
\beta_1 = \frac{T_i}{T_e}
\]

\[
\beta = \frac{4\gamma}{(1+3\gamma)}
\]

In equations (2.2), (2.3) and (2.4), the dust charge density $n_d$, dust particle velocity $u_d$, variable charge number of dust grain $Z_d$, space coordinate $x$ and time $t$ are normalized by $n_{d0}$, acoustic speed $c_d = \sqrt{(T_i Z_{d0}/m_d)}$, $Z_{d0}$, the Debye length $\lambda_D = \sqrt{(T_i/4\pi Z_{d0}n_{d0}e^2)}$ and the inverse of plasma frequency, $\omega_{pd}^{-1} = \sqrt{(m_d/4\pi n_{d0}Z_{d0}^2e^2)}$, respectively. The electrostatic potential $\phi$ is normalized by $T_i/e$ and $\gamma$ is the nonthermal parameter. We have followed the model of Cairns et al (1995a;1995b), where ions are assumed to follow nonthermal particle distribution. The parameter $\gamma$ in the distribution function $f(v)$ (Figure 1.1) defines the shape of the distribution and expresses the deviation from the Maxwellian state. This form of the distribution is convenient for the description of various observed particle distributions. For example, when $\gamma = 0$; we obtain a Maxwellian distribution and when $\gamma \rightarrow 1$, the distribution resembles two counter streaming beams with cold core distribution (Volosevich et al. (2002)).

The dust charge variable $Q_d$ is governed by the charge current balance equation (Melandso et al. (1993))

\[
\left( \frac{\partial}{\partial t} + u_d \frac{\partial}{\partial x} \right) Q_d = I_e + I_i
\]

It must be noted that the charging currents originate from the electrons and ions reaching the grain surface. The dust charging time is of the order of $10^{-9}$ s, while dust motion time is of the order of $10^{-3}$ s. Thus, the dust charge can quickly reach the local equilibrium at which the currents from the electrons and ions to the dust
grains are balanced. The current balance equation implies

\[ I_e + I_i \approx 0 \]  \hspace{1cm} (2.9)

We further suppose that the streaming velocities of electrons and ions are much smaller than the thermal velocities. The electron and ion currents are

\[ I_e = -e\pi r^2 \left( \frac{8T_e}{\pi m_e} \right)^{1/2} n_e \exp \left( \frac{e\Phi}{T_e} \right) \]

\[ I_i = e\pi r^2 \left( \frac{8T_i}{\pi m_i} \right)^{1/2} n_i \left( 1 - \frac{e\Phi}{T_i} \right) \]  \hspace{1cm} (2.10)

where \( \Phi \) denotes the dust grain surface potential relative to the plasma potential \( \phi \). From the current balance equation and following Xie et al. (1999), we have

\[ \tilde{\alpha}_1 \delta (1 + \beta\phi + \beta^2\phi^2) \exp(-\phi)(1 - \psi) - \exp(\beta_1(\phi + \psi)) = 0 \]  \hspace{1cm} (2.11)

where \( \psi = e\Phi/T_i \), \( \tilde{\alpha}_1 = \sqrt{\beta_1/\mu_i} \), \( \delta = n_{i0}/n_{e0} \) and the mass ratio of ions to electrons is given by \( \mu_i = m_i/m_e \simeq 1843 \). By using (2.11), we can calculate the dust charge \( Q_d = C\Phi \), where \( C \) is the capacitance of dust grain \( (C = r) \) and

\[ Z_d = \frac{\psi}{\psi_0} \]  \hspace{1cm} (2.12)

where \( \psi = \psi_0 \) at \( \phi = 0 \) is the dust surface floating potential with respect to the unperturbed plasma potential at an infinite distance from the dust grain surface.

In (2.11), by substituting \( \phi = 0 \) to determine the value of \( \psi_0 \), we have

\[ \tilde{\alpha}_1 \delta (1 - \psi_0) - \exp(\beta_1\psi_0) = 0 \]  \hspace{1cm} (2.13)

Now, by expanding \( \psi \) near \( \psi_0 \), \( Z_{d1} \) and \( Z_{d2} \) from (2.12) can be obtained as follows:

\[ Z_{d1} = \gamma_1 \phi_1, \quad \gamma_1 = \frac{\gamma_b}{\gamma_a \psi_0}, \]  \hspace{1cm} (2.14)

where

\[ \gamma_a = [\tilde{\alpha}_1 \delta + \beta_1 \exp(\beta_1\psi_0)] \]
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and

\[ \gamma_b = -[\beta_1 \exp(\beta_1 \psi_0) - \tilde{\alpha}_1 \delta (1 - \psi_0)(\beta - 1)]. \]

\[ Z_{d2} = \gamma_1 \phi_2 + \gamma_2 \phi_1^2, \quad \gamma_2 = \frac{\gamma_c}{\gamma_a \psi_0}, \quad (2.15) \]

with

\[ \gamma_c = \gamma_{c1} + \gamma_{c2} + \gamma_{c3}, \]

where

\[ \gamma_{c1} = \frac{1}{2}[\tilde{\alpha}_1 \delta (1 - \psi_0) - \beta_1^2 \exp(\beta_1 \psi_0)] \]

\[ \gamma_{c2} = \psi_0 \gamma_1 [\tilde{\alpha}_1 \delta (1 - \beta) - \beta_1^2 \exp(\beta_1 \psi_0)], \]

\[ \gamma_{c3} = -\frac{1}{2}(\psi_0 \gamma_1 \beta_1)^2 \exp(\beta_1 \psi_0). \]

The value of \( \psi_0 \) is always negative for the dust grains that are negatively charged, so the condition

\[ 1 \geq \tilde{\alpha}_1 \delta \quad (2.16) \]

must be satisfied.

2.3 Small amplitude solitary waves

A self-consistent dust charge variation with plasma potential would have considerable influence on the small amplitude as well as large amplitude solitary waves. Here we first study the small amplitude solitary waves with nonthermal ions. As mentioned earlier, reductive perturbation method will be used here. In order to investigate the properties of small amplitude dust-acoustic solitary waves, we construct a weakly nonlinear theory of dust-acoustic waves which leads to scaling of independent variable through the stretching coordinates as follows:

\[ \xi = \epsilon^{1/2}(x - v_0 t) \quad \text{and} \quad \tau = \epsilon^{3/2} t \]
where $\epsilon$ is a small parameter measuring the weakness of dispersion and $v_0$ is the solitary wave velocity. To strike an appropriate balance between nonlinearity and dispersion terms, the dependent variables $n_d$, $u_d$, $\phi$ and $Z_d$ are expanded from equilibrium values in the power of $\epsilon$ as

$$
\begin{bmatrix}
n_d \\
u_d \\
\phi \\
Z_d
\end{bmatrix} =
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix} + \sum_{r=1}^{\infty} \epsilon^r 
\begin{bmatrix}
n_{dr} \\
u_{dr} \\
\phi_r \\
Z_{dr}
\end{bmatrix}
$$

(2.17)

Using stretching coordinates and inserting equations (2.5), (2.6), (2.17) into (2.2-2.4), and equating the same powers of $\epsilon$, we obtain a set of equations as follows:

$$
n_{d1} = -\frac{\phi_1}{v_0}, \quad u_{d1} = -\frac{\phi_1}{v_0} \quad \text{and} \quad Z_{d1} = \gamma_1 \phi_1
$$

(2.18)

with

$$\frac{1}{v_0^2} = \gamma_1 + \frac{\beta_1 + \delta(1 - \beta)}{\delta - 1}.$$

To next order of $\epsilon$, we get

$$
v_0 \frac{\partial n_{d2}}{\partial \xi} - \frac{\partial n_{d1}}{\partial \tau} - \frac{\partial u_{d2}}{\partial \xi} - \frac{\partial (n_{d1} u_{d1})}{\partial \xi} = 0
$$

(2.19)

$$
v_0 \frac{\partial u_{d2}}{\partial \xi} - \frac{\partial u_{d1}}{\partial \tau} - u_{d1} \frac{\partial u_{d1}}{\partial \xi} = -\frac{\partial \phi_2}{\partial \xi} - Z_{d1} \frac{\partial \phi_1}{\partial \xi}
$$

(2.20)

$$
\frac{\partial^2 \phi_1}{\partial \xi^2} = n_{d2} + Z_{d1} n_{d1} + Z_{d2} + \phi_2 \left( \frac{n_{i0}}{Z_{d0}} \beta_1 + \frac{n_{e0}}{Z_{d0}} \frac{1 - \beta}{1 - \beta} \right)
+ \phi_1 \left( \frac{n_{i0}}{Z_{d0}} \beta_1^2 - \frac{n_{e0}}{Z_{d0}} \right).
$$

(2.21)

After some algebraic manipulations and eliminating all the second order terms, we obtain the following KdV equation

$$
\frac{\partial \phi_1}{\partial \tau} + A \phi_1 \frac{\partial \phi_1}{\partial \xi} + B \frac{\partial^3 \phi_1}{\partial \xi^3} = 0
$$

(2.22)
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where \( A = B\left(\frac{3\gamma_1 v_0^2}{v_0^2} - \frac{\beta_1^2 - \delta}{\delta - 1} - 2\gamma_2\right) \) and \( B = \frac{v_0^3}{2} \). Here \( A \) and \( B \) are the coefficients of nonlinear and dispersive terms, respectively. On introducing the new variable \( \chi = \xi - u_0 \tau \), \( u_0 \) is a constant velocity, the soliton solution of (2.22) in the stationary frame is given as

\[
\phi_1 = \phi_m \text{sech}^2 \left[ (\xi - u_0 \tau)/w \right],
\]

where \( \phi_m = 3u_0/A \) and \( w = 2\sqrt{B/u_0} \) are the soliton amplitude and width, respectively.

Apparent form of \( A \) suggests that it is function of a number of parameters such as \( \gamma_1, v_0, \beta_1, \gamma_2 \) etc. These parameters further depend on temperature of electrons and ions, nonthermality, relative density of ions to electrons and other entries characterizing dusty plasma in a complicated way. Thus, nonlinearity explicitly depends on a number of variables. However, dispersion is a simple function of \( v_0 \). Thus, we can strike a delicate balance between nonlinearity and dispersion for certain range of parameters to obtain soliton solution given by (2.23). We have performed such characterization by doing numerical computation of \( \phi_m \) as a function of relative density \( \delta (= \frac{n_i}{n_e}) \) and results are shown in figure 2.1. Other parameters have been taken as \( u_0 = 1 \) and \( \beta_1 = 0.001 \). Compressive solitons are observed up to certain range of \( \delta \) beyond which rarefactive solitons are obtained. The peak amplitude shifts with nonthermality and reverses its sign with increase in \( \delta \). Furthermore, there is a singularity for all values of \( \beta \). All these features are succinctly observed in figure 2.1.

2.4 Derivation of the modified KdV equation

The singularity occurs, when the coefficient of nonlinear term of (2.22) is zero, as shown in figure 2.1. For this case, KdV equation (2.22) is no longer valid. We derive
2.4. Derivation of the modified KdV equation

Figure 2.1: Variation of peak amplitude $\phi_m$ in KdV equation with $\delta (= \frac{n_{\infty}}{n_{0}})$ for different values of $\beta$ and with $u_0 = 1$ and $\beta_1 = 0.001$.

The modified KdV (mKdV) equation which governs the dynamics of small amplitude double layers when $A$ becomes zero. In such cases we must include higher order nonlinearity. For this purpose we include higher order nonlinear effects, which are important in the study of double layers. This demands the introduction of new stretching coordinates as follows:

$$\xi = \epsilon(x - v_0t) \quad \text{and} \quad \tau = \epsilon^3 t$$

where $\epsilon$ is small expansion parameter. Using the above stretching coordinates and substituting (2.17) into (2.2-2.4) and comparing the coefficients of $\epsilon^3$, we have

$$n_{d2} = \frac{u_{d2}}{v_0} + \frac{u_{d1}^2}{v_0^2}, \quad (2.24)$$

$$v_0 \frac{\partial u_{d2}}{\partial \xi} - u_{d1} \frac{\partial u_{d1}}{\partial \xi} = z_{d1} v_0 \frac{\partial u_{d1}}{\partial \xi} - \frac{\partial \phi_2}{\partial \xi}, \quad (2.25)$$
and

\[
\frac{\partial^2 \phi_1}{\partial \xi^2} = z_d n_d + z_d n_d + n_d + n_d + \phi_3 \left( \frac{n_{e0}}{z_d n_d} - \frac{n_{d0}}{z_d n_d} \beta - 1 \right) + \phi_3^3 \left( \frac{n_{e0}}{z_d n_d} \frac{1}{6} \frac{n_{d0}}{z_d n_d} \right) + \phi_1 \phi_2 \left( \frac{n_{e0}}{z_d n_d} \frac{\beta}{2} - \frac{n_{d0}}{z_d n_d} \beta \right).
\] (2.26)

Furthermore, comparing the coefficients of \( \epsilon^4 \), we obtain

\[
-v_0 \frac{\partial n_d}{\partial \tau} + \frac{\partial n_d}{\partial \xi} + \frac{\partial u_d}{\partial \xi} + \frac{3 u_d^2}{v_0^2} \frac{\partial u_d}{\partial \xi} + \frac{3}{2} \frac{u_d^2}{v_0^2} \frac{\partial u_d}{\partial \xi} = 0, \tag{2.27}
\]

\[
\frac{\partial u_d}{\partial \tau} - v_0 \frac{\partial u_d}{\partial \xi} + \frac{\partial (u_d u_d)}{\partial \xi} = \frac{\partial \phi_3}{\partial \xi} + z_d \frac{\partial \phi_2}{\partial \xi} + z_d \frac{\partial \phi_1}{\partial \xi}. \tag{2.28}
\]

After eliminating all the second and third order terms, we get the following mKdV equation:

\[
\frac{\partial \phi_1}{\partial \tau} + C \frac{\partial \phi_1}{\partial \xi} + A \frac{\partial (\phi_1 \phi_2)}{\partial \xi} + B \frac{\partial^3 \phi_1}{\partial \xi^3} = 0, \tag{2.29}
\]

where

\[
A = 3 \gamma_1 v_0 \frac{\gamma_1 v_0}{2} - \frac{3}{2} v_0 - \gamma_2 v_0^3 - B \left( \frac{n_{e0}}{n_{d0} z_d} \frac{\beta}{2} - \frac{n_{d0}}{n_{d0} z_d} \right), \tag{2.30}
\]

\[
B = \frac{v_0^3}{2}, \tag{2.31}
\]

\[
C = \frac{3}{2} \left[ \frac{5}{2} \gamma_1 v_0^2 + \gamma_2 v_0^3 + \frac{4}{3} \gamma_2 v_0 - \gamma_2 v_0^3 \right] - \frac{v_0^3}{3} \left( \frac{n_{e0}}{n_{d0} z_d} \frac{\beta}{2} + \frac{3 n_{d0}}{2 n_{d0} z_d} \left( \beta + \frac{1}{3} \right) \right). \tag{2.32}
\]

At A=0, the stationary solution of (2.29) can be written as

\[
\phi_1 = \phi_{1m} \text{sech} \left[ \frac{(\xi - u_0 \tau)}{L_1} \right], \tag{2.33}
\]

where \( \phi_{1m} = \sqrt{\frac{B}{v_0}} \) and \( L_1 = \sqrt{\frac{B}{v_0}} \) are the peak amplitude and width of soliton, respectively. Solitons exist when C>0.
2.5 Double layer solution

When $A \rightarrow 1$ but $A \neq 0$ and using $A\phi_2 \rightarrow D\phi_1^2$ (Xie et al. (1999)) in (2.29), we obtain the following equation

$$\frac{\partial \phi_1}{\partial \tau} + C\phi_1^2 \frac{\partial \phi_1}{\partial \xi} + D\phi_1 \frac{\partial \phi_1}{\partial \xi} + B \frac{\partial^3 \phi_1}{\partial \xi^3} = 0,$$

(2.34)

We present double layer solution associated with the mKdV equation. For this, we introduce a variable $\zeta = \xi - M\tau$ in a stationary frame, where M is a Mach number. Equation (2.34) on integration leads to the following equation

$$\frac{1}{2} \left( \frac{d\phi}{d\zeta} \right)^2 + V(\phi) = 0,$$

(2.35)

where $V(\phi)$ is the Sagdeev potential given by

$$V(\phi) = -\frac{M}{2} P\phi^2 + \frac{Q}{3} \phi^3 + \frac{R}{4} \phi^4.$$

(2.36)

Here

$$P = \frac{1}{B}, \quad Q = \frac{D}{2B}, \quad \text{and} \quad R = \frac{C}{3B}$$

(2.37)

For simplicity, we have introduced $\phi$ for $\phi_1$. While deriving (2.35), we have used the following boundary conditions

$$\phi \rightarrow 0, \quad \frac{\partial \phi}{\partial \zeta} \rightarrow 0, \quad \frac{\partial^2 \phi}{\partial \zeta^2} \rightarrow 0 \quad \text{as} \quad |\zeta| \rightarrow \infty.$$

(2.38)

Furthermore, (2.35) with (2.36) can be considered as equation of motion of a particle of unit mass under the action of the potential $V(\phi)$. They may also be regarded as the equations of anharmonic oscillation, provided $\zeta$ and $\phi$ are interpreted as time and space coordinates, respectively. For the DL solution, the Sagdeev potential should be negative between $\phi = 0$ and $\phi = \phi_{2m}$, where $\phi_{2m}$ is
some extreme value of potential. Additional boundary conditions that \( V(\phi) \) must satisfy for DL solution are as follows:

\[
\begin{align*}
V(\phi) &= 0 \quad \text{at} \quad \phi = 0 \quad \text{and} \quad \phi = \phi_{2m}, \\
V'(\phi) &= 0 \quad \text{at} \quad \phi = 0 \quad \text{and} \quad \phi = \phi_{2m}, \\
V''(\phi) &= 0 \quad \text{at} \quad \phi = 0 \quad \text{and} \quad \phi = \phi_{2m}.
\end{align*}
\] (2.39) (2.40) (2.41)

By using these boundary conditions in (2.36), we obtain

\[
M = -\frac{R^2 P}{2} \phi_{2m}^2, \quad \text{(2.42)}
\]

and

\[
\phi_{2m} = -\frac{2 Q}{3 R}. \quad \text{(2.43)}
\]

Using (2.42) and (2.43) in (2.36), we get

\[
V(\phi) = R^4 \phi^2 (\phi_{2m} - \phi)^2. \quad \text{(2.44)}
\]

The DL solution of (2.36) with (2.44) is given by

\[
\phi = \frac{\phi_{2m}}{2} [1 - \tanh \sqrt{-\frac{R}{8} \phi_{2m} (\xi - M\tau)}]. \quad \text{(2.45)}
\]

Equation (2.45) implies the existence of a DL whenever \( R \) is negative. Furthermore, this characterizes the nature of the double layers. The system will support compressive and rarefactive double layers depending on the sign of \( Q \). If \( Q \) is positive, a compressive double layer exists, whereas for negative \( Q \) a rarefactive double layer exists. The thickness of the double layer is given by

\[
d = 4 \left| \phi_{2m} \right| \left( -\frac{2}{R} \right)^{1/2}, \quad \text{(2.46)}
\]

Earlier investigations have shown that both compressive and rarefactive double layers are admissible in dusty plasma (Sakanaka and Shukla (2000)). We have
chosen the parameters as $u_0 = 1$ and $\beta_1 = 0.001$. Figure 2.2, displays $V(\phi)$ versus $\phi$ for the above chosen set of parameters, and apparently compressive double layers are obtained. The increase in nonthermal parameter decreases the amplitude as well as depth of potential well. Further increase in $\beta$ leads to disappearance of double layers.

Figures 2.3 and 2.4 display the peak amplitudes of solitons and double layers, respectively, as a function of $\delta(=\frac{n_m}{n_0})$ for three values of $\beta$. An interesting result is that in both cases an increase of $\delta$ leads to an increase of $\phi_{1m}$ and $\phi_{2m}$ respectively. However, width of DLs first increases with $\delta$ and subsequently falls with any further increase in $\delta$. The same behavior is observed for different $\beta$ but width decreases with an increase in nonthermality as shown in figure 2.5.
Chapter 2. Localized coherent nonlinear wave structures in dusty plasma with nonthermal ions

Figure 2.3: Variation of peak amplitude $\phi_{1m}$ in the mKdV with $\delta (= \frac{\nu_0}{nu_0})$ for different values of $\beta$ and with $u_0 = 1$ and $\beta_1 = 0.001$.

Figure 2.4: Variation of peak amplitude $\phi_{2m}$ in the DLs with $\delta (= \frac{\nu_0}{nu_0})$ for different values of $\beta$ and with $u_0 = 1$ and $\beta_1 = 0.0001$. 
2.6 Conclusions

In this chapter, we have studied the role of the nonthermal parameter $\beta$ both on dust-acoustic solitons and double layers in dusty plasma with variable dust charge and consisting of nonthermal ions. The KdV and mKdV equations are derived using reductive perturbation method. The results obtained have several ramifications as follows:

(i) It is concluded that when the variable charge is introduced, then there is possibility of both types of solitary waves for nonthermal distribution ($\beta \neq 0$). In case of nonthermal distribution, the range of $\delta$ for positive peak amplitude depends upon the nonthermal parameter $\beta$. It is noticed that there is possibility of both types of solitary waves for variable charge as compared with constant dust charge for which only one type of solitary waves are obtained (Mamun et al. (1996a)).
(ii) Only compressive DLs are observed. The nonthermal parameter $\beta$ play a crucial role for the existence of DLs.

(iii) It is emphasized that the findings of the present investigation may be useful in understanding the nonlinear structures of solitary waves in space and astrophysical environments where dust is the ubiquitous component.