Directed cycle Decompositions of Symmetric Digraphs
Chapter 3

Directed cycle Decompositions of Symmetric Digraphs

3.1 Introduction

As pointed out in the introduction, Cavenagh and Billington [28] have proved that the necessary conditions for the existence of a $C_5$-decomposition of the complete tripartite graph $K(r, s, t)$ are also sufficient when all the partite sets are even size. Further, Cavenagh and Billington [29] have studied the decomposition of complete multipartite graphs (not necessarily regular) into cycles of length 4, 6 and 8.

Billington et al. [22] solved the problem of decomposing $K_m \circ \overline{K}_n(\lambda)$ into 5-cycles. In [53], Manikandan and Paulraja proved that the necessary conditions for the existence of a $C_7$-decomposition of $K_m \circ \overline{K}_n$ are sufficient and they also proved in [54] that the necessary conditions for the existence of $C_p$-decomposition of $K_m \times K_n$ and $K_m \circ \overline{K}_n$ are sufficient, where $p \geq 11$.
is a prime. Further, Cavenagh [26] and Sotteau [66] has settled the existence of $C_k$-decomposition of complete tripartite graph and even regular complete $2k$-partite graph respectively. In [36], it has been shown that the complete $n$-partite regular graph can be decomposed into $k$-cycles if and only if both $n$ and $k$ are odd.

If $K_m \times K_n$ and $K_m \circ K_n$ are odd regular graph, then they cannot be decomposed into cycles of length $k$ even if $k$ divides the number of edges in $K_m \times K_n$ and $K_m \circ K_n$. Hence the problem of decomposing symmetric digraphs $(K_m \times K_n)^*$ and $(K_m \circ K_n)^*$ into directed cycles are considered, whenever the divisibility conditions are satisfied. Recently, Manikandan et al [55] proved that if $m \geq 3$, $t$ odd, $3 < t < mn$, $\gcd(m(m-1),t) \leq m$ and $t \mid m(m-1)n$, then $\overrightarrow{C}_t \mid (K_m \circ K_n)^*$, except possibly when $(t,m) = (3,6)$.

The above result leads to raise the following:

**Question:** Does there exist a $\overrightarrow{C}_{pq}$-decomposition in $(K_m \times K_n)^*$ for the odd primes $p,q \geq 3$?

In this chapter, the above question have been considered. In fact, it is proved that for $m, n \geq 3$ and $p = 5, 7$, $\overrightarrow{C}_{3p} \mid (K_m \times K_n)^*$ if and only if $3p \mid m(m-1)n(n-1)$ and $mn \geq 3p$. Further, it is shown that $\overrightarrow{C}_{pq} \mid (K_m \times K_n)^*$ for all primes $p,q \geq 3$ and $m, n \geq pq$. The Question 1 remains open for other values of $p, q$ and $m, n < pq$. Then problem of decomposing complete regular multipartite symmetric digraph into cycles of fixed length has been studied here. In fact, it is shown that, for a prime $p \in \{5, 7, 11, 13\}$, $\overrightarrow{C}_{3p} \mid (K_m \circ K_n)^*$ if and only if (i) $3p \mid m(m-1)n^2$ (ii) $m \geq 3$ and $mn \geq 3p$. 
3.2 Preliminary Results

Notation: Let

\[ V(K_m) = V(K_m^*) = \{v_0, v_1, \ldots, v_{m-1} \} \quad \text{and} \quad \]
\[ V(K_n) = \{w_0, w_1, \ldots, w_{n-1} \}. \]

Theorem 3.2.1. [2] \( K_{2n+1} \) can be decomposed into edge-disjoint Hamilton cycles.

Theorem 3.2.2. [66] If \( 2n \geq 8 \), then \( \overrightarrow{C}_{2n} \parallel K_{2n}^* \).

The proof of the results (Corollary 4.17.1 and 4.19.1) given in [58] make the following:

Theorem 3.2.3. [58] For \( m \geq 3 \), \( \overrightarrow{C}_k \mid (\overrightarrow{C}_k \times K_m^*) \).

Remark 3.2.4. The underlying graph of the graph in Theorem 3.2.3, gives \( C_k \mid C_k \times K_m \).

Theorem 3.2.5. [62] The digraph \( \overrightarrow{C}_r \circ K_m \) has a Hamilton decomposition.

Lemma 3.2.6. [74] For \( m, n \geq 3 \), \( \overrightarrow{C}_m \times \overrightarrow{C}_n \) can be decomposed into arc-disjoint directed Hamilton cycles, if and only if \( m \) or \( n \) is odd.

Note 3.1: For \( m \geq 3 \), \( \overrightarrow{C}_m \times \overrightarrow{C}_n \) is itself a Hamilton cycle if and only if \( m \) or \( n \) is odd.

Theorem 3.2.7. [59] \( C_m \mid C_m \circ K_n \), when \( m \geq 3, n \geq 1 \).

The construction of \( C_m \)-decompositions of \( C_m \circ K_n \) in [59] gives the following.
Corollary 3.2.8. $C_m \mid C_m \circ K_n$, when $m \geq 3, n \geq 1$. □

Theorem 3.2.9. [65] For odd $n$ (resp. even $n$), $C_k$ decomposes $K_n$. (resp. $C_k$ decomposes $K_n - I$, where $I$ is a 1-factor of $K_n$). The necessary conditions for the existence of $C_k$-decomposition when $n$ is odd (resp. even $n$) are
(i) $3 \leq k \leq n$ and
(ii) $2k \mid n(n - 1)$ (resp. $2k \mid n(n - 2)$). □

Theorem 3.2.10. [5] For positive integers $m$ and $n$, with $2 \leq m \leq n$, the digraph $K^*_n$ can be decomposed into directed cycles of length $m$ if and only if $m$ divides the number of arcs in $K^*_n$ and $m \neq (4,4), (6,3), (6,6)$. □

From the proof of the Theorem 2.7 in [26], we have the following:

Theorem 3.2.11. If $T^*_k$ is a directed closed trail of length $k$ with chromatic number $3$ and maximum degree at most $2l$, then $C_k \mid (T^*_k \circ K_l)$. □

Theorem 3.2.12. [6] If $d$ is an odd integer and $p$ is an odd prime, there is a 2-factorization of $K_d \circ K_p$ which consists entirely of $p$-cycles. □

Theorem 3.2.13. [55] If $m \geq 3$, $t$ is odd, $3 \leq t \leq mn$, $\gcd(m(m-1), t) \leq m$ and $t \mid m(m-1)n$, then $C_t \mid (K_m \circ K_n)^*$, except possibly when $(t, n) = (3, 6)$. □

Theorem 3.2.14. [1] If $S = \{3, 5, 6, 8, 10\}$, then there exists a PBD$(v, S)$ for any integer $v \geq 3$ with genuine exceptions $v \in \{4, 12, 14, 20\}$ implies that $M \mid K_v$ holds for some subset $M \subseteq \{K_3, K_5, K_6, K_8, K_{10}\}$. □
3.3 \( \vec{C}_{15} \)-Decomposition of \( (K_t \times K_n)^* \), \( t \in \{6, 10\} \)

In this section, some basic constructions for the existence of \( \vec{C}_{15} \)-decompositions of symmetric digraphs required in the proof of our main result are presented.

Remark 3.3.1. \([31,32]\) \( C_3 \mid K_{2,2,2} \). □

Lemma 3.3.2. \( C_{15} \mid (C_5 \times K_6) \).

Proof. Let \( V(C_5 \times K_6) = \bigcup_{i=0}^{4} V_i \), where \( V_i = \{v_i^0, \ldots, v_i^5\} \). Partition \( V_i, 0 \leq i \leq 4 \), into 2-element subsets and identify each 2-element subset into a single vertex. Obtain a new graph \( G \) with vertex set as the identified vertices and join two of them by an edge if and only if the corresponding 2-element subsets induces a \( K_{2,2} \) or \( K_2 \times K_2 \). Then the resulting graph \( G \cong C_5 \times K_3 \oplus 3C_5 \). Now \( C_{15} \)-decomposition of \( C_5 \times K_3 \) are given as follows.

Let

\[
H_1 = \alpha_1(V_1, V_2) \oplus \alpha_1(V_2, V_3) \oplus \alpha_2(V_3, V_4) \oplus \alpha_1(V_4, V_5) \oplus \alpha_2(V_5, V_1),
\]

\[
H_2 = \alpha_2(V_1, V_2) \oplus \alpha_2(V_2, V_3) \oplus \alpha_1(V_3, V_4) \oplus \alpha_2(V_4, V_5) \oplus \alpha_1(V_5, V_1).
\]

be the edge-disjoint cycles of length 15. By blowing up the vertices of \( G \) into 2-element subsets, each \( H_i, i = 1, 2, \) in the \( C_{15} \)-decomposition of \( C_5 \times K_3 \) in \( G \) becomes a graph isomorphic to \( C_{15} \circ K_2 \) in \( C_5 \times K_6 \) and \( 3C_5 \in G \) becomes three 10-cycles, say \( \{H_i', i = 1, 2, 3\} \), where

\[
H_1' = (v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_0^1 v_1^1 v_2^1 v_3^1 v_4^1),
\]

\[
H_2' = (v_0^2 v_1^2 v_2^2 v_3^2 v_4^2 v_0^3 v_1^3 v_2^3 v_3^3),
\]
By Theorem 3.2.7, $C_{15} \circ K_2$. Now consider $H_2 \oplus \{H'_1 \oplus H'_2 \oplus H'_3\}$ and decompose it into six $C_{15}$ as follows. Let

\[
H'_{2,1} = (v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_1^1 v_2^1 v_3^1 v_4^1 v_5^1),
\]
\[
H'_{2,2} = (v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0),
\]
\[
H'_{2,3} = (v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0),
\]
\[
H'_{2,4} = (v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0),
\]
\[
H'_{2,5} = (v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0),
\]
\[
H'_{2,6} = (v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0).
\]

Clearly, $\{H_{2,1}, \ldots, H_{2,6}\}$ is a $C_{15}$-decomposition of $H_2 \oplus \{H'_1 \oplus H'_2 \oplus H'_3\}$.

Hence $C_{15} \mid C_5 \times K_6$, see Figure 3.1. □

Corollary 3.3.3. $\overrightarrow{C}_{15} \mid (C_5 \times K_6)^*$. □

Lemma 3.3.4. $\overrightarrow{C}_{15} \mid \overrightarrow{C}_5 \times K_6^*.$

Proof. As $\overrightarrow{C}_5 \times K_6^* \cong K_6^* \times \overrightarrow{C}_5$, consider $K_6^* \times \overrightarrow{C}_5$.

Now $V(K_6^* \times \overrightarrow{C}_5) = \bigcup_{i=0}^{5} V_i$, where $V_i = \{v_0^i, v_1^i, v_2^i, v_3^i, v_4^i\}$, $0 \leq i \leq 5$.

First decompose $K_6^*$ into $\{\overrightarrow{C}_3, \overrightarrow{T}_{15}\}$ as follows: Let

\[
\overrightarrow{H}_1 = (v_0 v_1 v_5),
\]
\[
\overrightarrow{H}_2 = (v_0 v_3 v_4),
\]
\[
\overrightarrow{H}_3 = (v_1 v_3 v_2),
\]
\[
\overrightarrow{H}_4 = (v_5 v_2 v_3),
\]
\[
\overrightarrow{H}_5 = (v_3 v_1 v_4)
\]
and
\[
\overrightarrow{T}_{15} = (v_0v_4v_1v_2v_3v_5v_0v_3v_4v_2v_4v_5v_1).
\]

Clearly, \(\{\overrightarrow{H}_1, \overrightarrow{H}_2, \overrightarrow{H}_3, \overrightarrow{H}_4, \overrightarrow{H}_5, \overrightarrow{T}_{15}\}\) is a \(\{\overrightarrow{C}_3, \overrightarrow{T}_{15}\}\)-decomposition of \(K_6\), see Figure 3.2.

Figure 3.1: \(\overrightarrow{C}_{15}\)-decomposition of \((H_1 \oplus \{H_1^1 \oplus H_2^1 \oplus H_3^1\})\)

Figure 3.2: \(C_{15}\)-decomposition of \((H_1 \oplus \{H_1^1 \oplus H_2^1 \oplus H_3^1\})\)
Figure 3.1: $C_{15}$-decomposition of $(H_1 \oplus (H_1^1 \oplus H_1^2 \oplus H_1^3))$
Figure 3.2: $\{\overrightarrow{C}_3, \overrightarrow{T}_{15}\}$-decomposition of $K^*_6$. 
Now write
\[ K_6^* \times \overrightarrow{C}_5 = ((\oplus \overrightarrow{C}_3) \oplus \overrightarrow{T}_{15}) \times \overrightarrow{C}_5 \]
\[ = \oplus(\overrightarrow{C}_3 \times \overrightarrow{C}_5) \oplus (\overrightarrow{T}_{15} \times \overrightarrow{C}_5) \]
\[ = (\oplus \overrightarrow{C}_{15}) \oplus (\overrightarrow{T}_{15} \times \overrightarrow{C}_5), \text{ by Note 3.1.} \]

It remains to show that \( \overrightarrow{C}_{15} \mid \overrightarrow{T}_{15} \times \overrightarrow{C}_5 \). Let
\[ \overrightarrow{H} = (v_0^0 v_1^1 v_2^2 v_3^3 v_4^4 v_5^3) \] be the base cycle of \( \overrightarrow{T}_{15} \times \overrightarrow{C}_5 \) and let \( \rho \) be the permutation \((V_0)(V_1)(V_2)(V_3)(V_4)(V_5)\). Then \( H, \rho(H), \rho^2(H), \rho^3(H), \rho^4(H) \) is a \( \overrightarrow{C}_{15} \)-decomposition of \( \overrightarrow{T}_{15} \times \overrightarrow{C}_5 \), see Figure 3.3. Thus \( \overrightarrow{C}_{15} \mid \overrightarrow{T}_{15} \times \overrightarrow{C}_5 \) and hence \( \overrightarrow{C}_{15} \mid \overrightarrow{C}_5 \times K_6^* \). \( \square \)
Lemma 3.3.5. \( \{ \overrightarrow{C}_3, \overrightarrow{C}_5 \} \mid K_8^* \).  

**Proof.** Let \( V(K_8) = \{1, 2, 3, 1', 2', 3', a, b\} \). Now construct \( \{ \overrightarrow{C}_3, \overrightarrow{C}_5 \} \) decompositions of \( K_8^* \) as follows: Clearly \((1, 2, 3), (1', 2', 3'), (1', 3', 2'), (a, 1', 2'), (a, 2', 1), (a, 2, 3'), (a, 3', 2'), (1', 3, b), (3, a, b), (1', a, 3), (1', b, a)\) are arc-disjoint directed 3-cycles of \( K_8^* \) and \((1, 1', 2, 2', b), (2', 3, 3', 1, b), (3', 3, 2', 2, b), (2', 1', 1', 3', b)\) are arc-disjoint directed 5-cycles of \( K_8^* \). Thus \( \{ \overrightarrow{C}_3, \overrightarrow{C}_5 \} \mid K_8^* \). \(\square\)

**Remark 3.3.6.** \([27] \overrightarrow{C}_5 \mid K_{2,2,4} \). \(\square\)

**Lemma 3.3.7.** For \( r = 4, 12 \), \( \overrightarrow{C}_{15} \mid (K_t \times K_r)^* \), \( t \in \{6, 10\} \). \(\square\)

**Lemma 3.3.8.** \( \overrightarrow{C}_{15} \mid (K_6 \times K_6)^* \). \(\square\)

**Lemma 3.3.9.** \( \overrightarrow{C}_{15} \mid (K_8 \times K_6)^* \). \(\square\)

**Lemma 3.3.10.** \( \overrightarrow{C}_{15} \mid (K_8 \times K_{10})^* \). \(\square\)

Proof of the Lemmas 3.3.7 to 3.3.10 follows from Note 3.1, Theorem 3.2.10 and Lemmas 3.3.4 and 3.3.5.

**Lemma 3.3.11.** For all even \( r > 4 \), \( \{ K_r, K_{r+2}, C_3, C_5 \} \mid K_{3r+2} \).  

**Proof.** Now write \( K_{3r+2} = 2K_r \oplus K_{r+2} \oplus K_{r,r+2} \).

Let \( V_1, V_2, V_3 \) be the partite sets of \( K_{r,r+2} = G \), where
\[
V_1 = \{ v_1^1, \ldots, v_1^r \},
\]
\[
V_2 = \{ v_2^1, \ldots, v_2^r \} \quad \text{and}
\]
\[
V_3 = \{ v_3^1, \ldots, v_3^{r+2} \}.
\]
Now partition the vertices of each $V_i$ into 2-element subsets and identify them into a single vertex except the last 2-element subset of $V_3$. Obtain a new 3-partite graph $G'$ with vertex set; as the identified vertices of $V_i$ and join two of them by an edge if the corresponding 2-element subsets form a $K_{2,2}$ in $G$. The new graph

$$G' \cong K_{r/2,r/2,r/2}$$

$$= (C_3 \Box \overline{K}_{r/2}) \oplus (C_3 \times K_{r/2})$$

$$= r/2(C_3) \oplus (\oplus C_3), \text{by Remark 3.2.4.}$$

When blowing up the vertices of $G'$, each $C_3$ in the $C_3$-decomposition of $C_3 \times K_{r/2}$ is isomorphic to $K_{2,2,2}$ in $G$ and $C_3 \mid K_{2,2,2}$ by Remark 3.3.1 and the subgraph of of $G$ correspond to each $C_3$ in $C_3 \Box \overline{K}_{r/2}$ together with the edges between that subgraph and last two vertices $v_{r+1}^3$ and $v_{r+2}^3$ of $V_3$ forms a graph isomorphic to $K_{2,2,4}$ in $G$. So by the above process, corresponding to each $C_3$ in $C_3 \Box \overline{K}_{r/2}$ we get a subgraph isomorphic to $K_{2,2,4}$ in $G$. As there are $r/2$ such $C_3$ in $C_3 \Box \overline{K}_{r/2}$, we get $r/2$ subgraphs isomorphic to $K_{2,2,4}$ and are edge-disjoint. By Remark 3.3.6, $C_5 \mid K_{2,2,4}$. This shows that $\{C_3, C_5\} \mid K_{r,r+2}$ and hence $\{K_r, K_{r+2}, C_3, C_5\} \mid K_{3r+2}$ for all even $r > 4$. □

**Lemma 3.3.12.** $C_{15} \mid (K_{14} \times K_6)^*$.

**Proof.** $K_{14} = 2K_4 \oplus K_6 \oplus 2K_{2,2,4} \oplus 2K_{2,2,2}$. Therefore,

$$(K_{14} \times K_6)^* = (2K_4^* \oplus K_6^* \oplus 2K_{2,2,4}^* \oplus 2K_{2,2,2}^*) \times K_6^*$$

$$= 2(K_4^* \times K_6^*) \oplus (K_6^* \times K_6^*) \oplus 2(K_{2,2,4}^* \times K_6^*) \oplus 2(K_{2,2,2}^* \times K_6^*)$$
by Theorem 3.2.10, Remarks 3.3.1 and 3.3.6. \( \overrightarrow{C_{15}} \)-decomposition of the graphs in the R.H.S of the equation 3.1 exist by Note 3.1 and Lemma 3.3.4. Thus \( \overrightarrow{C_{15}} \mid (K_{14} \times K_6)^* \).

Similarly, we obtain the following:

Lemma 3.3.13. \( \overrightarrow{C_{15}} \mid (K_{14} \times K_{10})^* \).

Lemma 3.3.14. \( \overrightarrow{C_{15}} \mid (K_{20} \times K_6)^* \).

Lemma 3.3.15. \( \overrightarrow{C_{15}} \mid (K_{20} \times K_{10})^* \).

Proof of the Lemmas 3.3.13, 3.3.14 and 3.3.15 follows from Note 3.1, Corollary 3.3.3 and Lemmas 3.3.8, 3.3.9 and 3.3.11.

Theorem 3.3.16. For all \( n \geq 3 \), \( \overrightarrow{C_{15}} \mid (K_t \times K_n)^* \), for \( t \in \{6,10\} \).

Proof. For \( n \notin \{4,12,14,20\} \), \( (K_t \times K_n)^* = \oplus \overrightarrow{C_{15}}, \) by Note 3.1, Theorem 3.2.10 and 3.2.14, as \( \overrightarrow{C_{15}} \mid (K_5 \times K_n)^* \), by Lemmas 3.3.9 and 3.3.10 and \( \overrightarrow{C_{15}} \mid (\overrightarrow{C_5} \times K_n)^* \), by Lemma 3.3.4.

When \( n \in \{4,12,14,20\} \), \( \overrightarrow{C_{15}} \mid (K_t \times K_n)^* \), by Lemmas 3.3.7, 3.3.12, 3.3.13, 3.3.14 and 3.3.15. Thus \( \overrightarrow{C_{15}} \mid (K_t \times K_n)^* \), for all \( n \geq 3 \) and \( t \in \{6,10\} \).

3.4 \( \overrightarrow{C_{21}} \)-Decomposition of \( (K_t \times K_n)^* \), \( t \in \{7,15\} \)

Some basic constructions for the existence of \( \overrightarrow{C_{21}} \)-decomposition of symmetric digraphs are presented here, which are required in the proof of our main result.
Lemma 3.4.1. $\overrightarrow{T}_{21} | K_7$.

**Proof.** Decompose $K_7$ into trails of length 21 as follows. Let

\[
\overrightarrow{H}_1 = (v_0v_5v_6v_7v_1v_4v_4v_4v_4v_6v_2v_1v_2v_5v_6v_0v_1v_0v_2v_0v_3v_0),
\overrightarrow{H}_2 = (v_3v_1v_3v_2v_3v_4v_3v_3v_5v_2v_3v_2v_1v_2v_3v_4v_3v_3v_0).
\]

Clearly, $\{\overrightarrow{H}_1, \overrightarrow{H}_2\}$ is a $\overrightarrow{T}_{21}$-decomposition of $K_7$, see Figure 3.4. □

Lemma 3.4.2. $\overrightarrow{T}_{21} | K_{15}$.

**Proof.** Decompose $K_{15}$ into trails of length 21 as follows. Let

\[
\overrightarrow{H}_1 = (v_0v_6v_7v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4),
\overrightarrow{H}_2 = (v_3v_0v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4),
\overrightarrow{H}_3 = (v_1v_7v_8v_9v_{10}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4),
\overrightarrow{H}_4 = (v_1v_7v_8v_9v_{10}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4),
\overrightarrow{H}_5 = (v_1v_7v_8v_9v_{10}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4),
\overrightarrow{H}_6 = (v_1v_7v_8v_9v_{10}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4),
\overrightarrow{H}_7 = (v_1v_7v_8v_9v_{10}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4),
\overrightarrow{H}_8 = (v_1v_7v_8v_9v_{10}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4),
\overrightarrow{H}_9 = (v_1v_7v_8v_9v_{10}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4),
\overrightarrow{H}_{10} = (v_1v_7v_8v_9v_{10}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4v_5v_6v_7v_8v_9v_{10}v_3v_4).
\]

Clearly, $\{\overrightarrow{H}_1, ..., \overrightarrow{H}_{10}\}$ is a $\overrightarrow{T}_{21}$-decomposition of $K_{15}$. □
Figure 3.4: $\overrightarrow{T}_{21}$-decomposition of $K_7^*$. 
Lemma 3.4.3. \( \overline{C}_{21} \mid (K^*_7 \times \overline{C}_5) \).

Proof. Let \( V(K^*_7 \times \overline{C}_5) = \bigcup_{i=0}^{6} V_i \), where \( V_i = \{ v_i^0, v_i^1, v_i^2, v_i^3, v_i^4 \} \),
\( 0 \leq i \leq 6 \). By Lemma 3.4.1, we write \( K^*_7 = \overline{H}_1 \oplus \overline{H}_2 \), where \( \overline{H}_i \) is a directed trail of length 21. Therefore

\[
K^*_7 \times \overline{C}_5 = (\overline{H}_1 \oplus \overline{H}_2) \times \overline{C}_5
= (\overline{H}_1 \times \overline{C}_5) \oplus (\overline{H}_2 \times \overline{C}_5).
\]

Now \( \overline{C}_{21} \)-decomposition of \( \overline{H}_i \times \overline{C}_5 \), for \( i = 1, 2 \) is as follows. Let \( \overline{H}_1 = (v_0^0v_1^2v_2^3v_3^4v_4^0v_5^1v_6^2v_7^3v_8^4v_9^0v_10^1v_11^2v_12^3v_13^4v_14^0) \), be the base cycle of \( H_1 \times \overline{C}_5 \). Let the permutation \( \rho \) be defined as

\[
\rho = (V_1)(V_2)(V_3)(V_4)(V_5)(V_6)(v_0^2)(v_1^3)(v_2^4)(v_3^0),
\]
where \( (V_i) = (v_i^0v_i^1v_i^2v_i^3v_i^4) \). Then \( \{ \rho^i(\overline{H}_1) \mid 0 \leq i \leq 4 \} \) is a required \( \overline{C}_{21} \)-decomposition of \( H_1 \times \overline{C}_5 \).

Let \( \overline{H}_2 = (v_0^0v_1^2v_2^3v_3^4v_4^0v_5^1v_6^2v_7^3v_8^4v_9^0v_10^1v_11^2v_12^3v_13^4v_14^0) \), be the base cycle of \( H_2 \times \overline{C}_5 \). Let the permutation \( \rho \) be defined as

\[
\rho = (V_1)(V_2)...(V_6)(v_0^2)(v_1^3)(v_2^4)(v_3^0),
\]
where \( (V_i) = (v_i^0v_i^1v_i^2v_i^3v_i^4) \). Then \( \{ \rho^i(\overline{H}_2) \mid 0 \leq i \leq 4 \} \) is a required \( \overline{C}_{21} \)-decomposition of \( H_2 \times \overline{C}_5 \). Hence \( \overline{C}_{21} \mid (K^*_7 \times \overline{C}_5) \). \( \square \)

Lemma 3.4.4. \( \overline{C}_{21} \mid (K^*_{15} \times \overline{C}_5) \).

Proof. Let \( V(K^*_{15} \times \overline{C}_5) = \bigcup_{i=0}^{14} V_i \), where \( V_i = \{ v_i^0, v_i^1, v_i^2, v_i^3, v_i^4 \} \), \( 0 \leq i \leq 14 \).
Now write \( K^*_{15} = \overline{H}_1 \oplus \cdots \oplus \overline{H}_{10} \), where \( \overline{H}_i \) is a directed trail of length 21, by Lemma 3.4.2. Therefore

\[
K^*_{15} \times \overline{C}_5 = (\overline{H}_1 \oplus \cdots \oplus \overline{H}_{10}) \times \overline{C}_5
\]
Now $\overline{C}_{21}$-decomposition of $H_i \times \overline{C}_5$, for $i = 1, \ldots, 10$ are constructed as follows.

Let $H_1 = (v^0_0 v_1^1 v_2^2 v_3^3 v_4^4 v_5^5)$, be the base cycle of $H_1 \times \overline{C}_5$. Let the permutation $\rho$ be defined as

$$\rho = (V_0)(V_1)(V_2)(V_3)(V_4) \cdots (V_{13})(V_{14})(V_{15}),$$

where $(V_i) = (v^0_i v_1^1 v_2^2 v_3^3 v_4^4)$. Then $\{\rho'(H_1) \mid 0 \leq i \leq 4\}$ is a required $\overline{C}_{21}$-decomposition of $H_1 \times \overline{C}_5$.

Let $H_2 = (v^0_0 v_1^1 v_2^2 v_3^3 v_4^4 v_5^5)$, be the base cycle of $H_2 \times \overline{C}_5$. Let the permutation $\rho$ be defined as

$$\rho = (V_0)(V_1)(V_2)(V_3)(V_4) \cdots (V_{13})(V_{14})(V_{15}),$$

where $(V_i) = (v^0_i v_1^1 v_2^2 v_3^3 v_4^4)$. Then $\{\rho'(H_2) \mid 0 \leq i \leq 4\}$ is a required $\overline{C}_{21}$-decomposition of $H_2 \times \overline{C}_5$.

Let $H_3 = (v^0_0 v_1^1 v_2^2 v_3^3 v_4^4 v_5^5)$, be the base cycle of $H_3 \times \overline{C}_5$. Let the permutation $\rho$ be defined as

$$\rho = (V_0)(V_1)(V_2)(V_3)(V_4) \cdots (V_{13})(V_{14})(V_{15}),$$

where $(V_i) = (v^0_i v_1^1 v_2^2 v_3^3 v_4^4)$. Then $\{\rho'(H_3) \mid 0 \leq i \leq 4\}$ is a required $\overline{C}_{21}$-decomposition of $H_3 \times \overline{C}_5$.

Let $H_4 = (v^0_0 v_1^1 v_2^2 v_3^3 v_4^4 v_5^5)$, be the base cycle of $H_4 \times \overline{C}_5$. Let the permutation $\rho$ be defined as

$$\rho = (V_0)(V_1)(V_2)(V_3)(V_4) \cdots (V_{13})(V_{14})(V_{15}),$$

where $(V_i) = (v^0_i v_1^1 v_2^2 v_3^3 v_4^4)$. Then $\{\rho'(H_4) \mid 0 \leq i \leq 4\}$ is a required $\overline{C}_{21}$-decomposition of $H_4 \times \overline{C}_5$. 

Now $\overline{C}_{21}$-decomposition of $H_i \times \overline{C}_5$, for $i = 1, \ldots, 10$ are constructed as follows.

Let $H_1 = (v^0_0 v_1^1 v_2^2 v_3^3 v_4^4 v_5^5)$, be the base cycle of $H_1 \times \overline{C}_5$. Let the permutation $\rho$ be defined as

$$\rho = (V_0)(V_1)(V_2)(V_3)(V_4) \cdots (V_{13})(V_{14})(V_{15}),$$

where $(V_i) = (v^0_i v_1^1 v_2^2 v_3^3 v_4^4)$. Then $\{\rho'(H_1) \mid 0 \leq i \leq 4\}$ is a required $\overline{C}_{21}$-decomposition of $H_1 \times \overline{C}_5$.
where \((V_i) = (v_i^0v_i^1v_i^2v_i^3v_i^4).\) Then \(\{\rho^i(\vec{H}_i) \mid 0 \leq i \leq 4\}\) is a required \(\vec{C}_{21}\)-decomposition of \(H_4 \times \vec{C}_5\).

Let \(\vec{H}_5 = (v_i^0v_i^1v_i^2v_i^3v_i^4v_i^5v_i^6v_i^7v_i^8v_i^9v_i^{10}),\)
be the base cycle of \(H_5 \times \vec{C}_5\). Let the permutation \(\rho\) be defined as

\[
\rho = (V_0)(V_1)\cdots(V_9)(V_{11})(V_{12})(V_{13})(V_{14})(v_i^0)(v_i^1)(v_i^2)(v_i^3)(v_i^4),
\]
where \((V_i) = (v_i^0v_i^1v_i^2v_i^3v_i^4).\) Then \(\{\rho^i(\vec{H}_5) \mid 0 \leq i \leq 4\}\) is a required \(\vec{C}_{21}\)-decomposition of \(H_5 \times \vec{C}_5\).

Let \(\vec{H}_6 = (v_i^0v_i^1v_i^2v_i^3v_i^4v_i^5v_i^6v_i^7v_i^8v_i^9v_i^{10}),\)
be the base cycle of \(H_6 \times \vec{C}_5\). Let the permutation \(\rho\) be defined as

\[
\rho = (V_0)(V_1)\cdots(V_9)(V_{11})(V_{12})(V_{13})(V_{14})(v_i^0)(v_i^1)(v_i^2)(v_i^3)(v_i^4),
\]
where \((V_i) = (v_i^0v_i^1v_i^2v_i^3v_i^4).\) Then \(\{\rho^i(\vec{H}_6) \mid 0 \leq i \leq 4\}\) is a required \(\vec{C}_{21}\)-decomposition of \(H_6 \times \vec{C}_5\).

Let \(\vec{H}_7 = (v_i^0v_i^1v_i^2v_i^3v_i^4v_i^5v_i^6v_i^7v_i^8v_i^9v_i^{10}),\)
be the base cycle of \(H_7 \times \vec{C}_5\). Let the permutation \(\rho\) be defined as

\[
\rho = (V_0)(V_1)\cdots(V_9)(V_{11})(V_{12})(V_{13})(V_{14})(v_i^0)(v_i^1)(v_i^2)(v_i^3)(v_i^4),
\]
where \((V_i) = (v_i^0v_i^1v_i^2v_i^3v_i^4).\) Then \(\{\rho^i(\vec{H}_7) \mid 0 \leq i \leq 4\}\) is a required \(\vec{C}_{21}\)-decomposition of \(H_7 \times \vec{C}_5\).
Let \( \overrightarrow{H}_9 = (v_1^0 v_2 v_3^2 v_4^2 v_5^2 v_6 v_7 v_8 v_9 v_{10}) \),
be the base cycle of \( H_5 \times \overrightarrow{G}_5 \). Let the permutation \( \rho \) be defined as
\[
\rho = (V_0)(V_1) \cdots (V_9)(V_{10})(V_{11})(V_{12})(V_{13})(V_{14})(V'_{10})(V'_{11})(V'_{12})(V'_{13})(V'_{14})(V'_{15}),
\]
where \( (V_i) = (v_i^0 v_i v_i^3 v_i^4) \). Then \( \left\{ \rho' (\overrightarrow{H}_9) \mid 0 \leq i \leq 4 \right\} \) is a required
\( \overrightarrow{C}_{21} \)-decomposition of \( H_9 \times \overrightarrow{G}_5 \).

Let \( \overrightarrow{H}_{10} = (v_8^0 v_9^2 v_9^3 v_9^4 v_9^5 v_9^6 v_9^7 v_9^8 v_9^9 v_9^0 v_9^1 v_9^2 v_9^3 v_9^4 v_9^5 v_9^6 v_9^7 v_9^8 v_9^9),\)
be the base cycle of \( H_7 \times \overrightarrow{G}_5 \). Let the permutation \( \rho \) be defined as
\[
\rho = (V_0)(V_1) \cdots (V_8)(V_9)(V_{10}) \cdots (V_{14})(V'_{10})(V'_{11})(V'_{12})(V'_{13})(V'_{14})(V'_{15}),
\]
where \( (V_i) = (v_i^0 v_i v_i^3 v_i^4) \). Then \( \left\{ \rho' (\overrightarrow{H}_{10}) \mid 0 \leq i \leq 4 \right\} \) is a required
\( \overrightarrow{C}_{21} \)-decomposition of \( H_{10} \times \overrightarrow{G}_5 \). Hence \( \overrightarrow{C}_{21} \mid (K_{10}^* \times \overrightarrow{G}_5) \).

**Lemma 3.4.5.** For \( r = 4, 12 \), \( \overrightarrow{C}_{21} \mid (K_r^* \times K_r)^* \), where \( t \in \{7, 15\} \).

**Proof.** Now write
\[
(K_r^* \times K_r)^* = K_r^* \times K_r^* = (\hat{\overrightarrow{C}}_7) \times (\hat{\overrightarrow{C}}_3), \text{ by Theorem 3.2.10}
\]
\[
= (\hat{\overrightarrow{C}}_7 \times \overrightarrow{C}_3)
\]
\[
= \hat{\overrightarrow{C}}_{21}, \text{ by Note 3.1.}
\]

**Lemma 3.4.6.** For \( t = 7, 15 \), \( \overrightarrow{C}_{15} \mid (K_t^* \times K_t)^* \).

**Proof.** Now write
\[
(K_t^* \times K_t)^* = K_t^* \times K_t^* = (\overrightarrow{C}_3 \oplus \cdots \oplus \overrightarrow{C}_3) \times K_t^*, \text{ by Lemma 3.3.5}
\]
Lemma 3.4.7. For \( t = 7, 15 \), \( C_{15} \mid (K_{14} \times K_t)^* \).

Proof. Now \( K_{14} = 2K_4 \oplus K_6 \oplus 8C_3 \oplus 8C_5 \), by Lemma 3.3.11. Therefore

\[
(K_{14} \times K_t)^* = (2K_4 \oplus K_6 \oplus 8C_3 \oplus 8C_5)^* \times K_t^*
\]

\[
= 2(K_4^* \times K_t^*) \oplus (K_6^* \times K_t^*) \oplus 8(C_3^* \times K_t^*) \oplus 8(C_5^* \times K_t^*)
\]

\[
= \oplus(C_3 \times C_7) \oplus (C_5 \times K_t^*) \oplus 16(C_3 \times C_7) \oplus 16(C_5 \times K_t^*)
\]

\[
= \oplus C_{21}, \text{ by Note 3.1, Theorem 3.2.10 and Lemmas 3.4.3, 3.4.4.} \Box
\]

Lemma 3.4.8. For \( t = 7, 15 \), \( C_{15} \mid (K_{20} \times K_t)^* \).

Proof. Now \( K_{20} = 2K_6 \oplus K_8 \oplus 4C_3 \oplus 24C_5 \), by Lemma 3.3.11. Therefore

\[
(K_{20} \times K_t)^* = (2K_6 \oplus K_8 \oplus 4C_3 \oplus 24C_5)^* \times K_t^*
\]

\[
= 2(K_6^* \times K_t^*) \oplus (K_8^* \times K_t^*) \oplus 8(C_3^* \times K_t^*) \oplus 24(C_5^* \times K_t^*)
\]

\[
= \oplus(C_5 \times C_7) \oplus (K_8^* \times K_t^*) \oplus 8(C_3 \times C_7) \oplus 48(C_5 \times K_t^*)
\]

\[
= \oplus C_{21}, \text{ by Note 3.1, Theorem 3.2.10 and Lemmas 3.4.3, 3.4.4, 3.4.6.} \Box
\]

Theorem 3.4.9. For all \( n \geq 3 \), \( C_{21} \mid (K_t \times K_n)^* \), where \( t \in \{7, 15\} \).

Proof. For \( n \notin \{4, 12, 14, 20\} \), \((K_t \times K_n)^* = \oplus C_{21}\), by Note 3.1, Theorems 3.2.10 and 3.2.14, as \( C_{21} \mid (K_8 \times K_n)^* \), by Lemma 3.4.6 and \( C_{21} \mid (C_5 \times K_n)^* \),
by Lemmas 3.4.3 and 3.4.4.

When \( n \in \{4, 12, 14, 20\} \), \( \overrightarrow{C}_{21} | (K_t \times K_n)^* \), by Lemmas 3.4.5, 3.4.7 and 3.4.8. Thus \( \overrightarrow{C}_{21} | (K_t \times K_n)^* \), for \( t \in \{7, 15\} \). □

Lemma 3.4.10. \( \overrightarrow{C}_{21} | \overrightarrow{C}_7 \times K_6^* \).

**Proof.** As \( \overrightarrow{C}_7 \times K_6^* \cong K_5^* \times \overrightarrow{C}_7 \), consider \( K_6^* \times \overrightarrow{C}_7 \).

Now \( V(K_6^* \times \overrightarrow{C}_7) = \bigcup_{i=0}^{5} V_i \), where \( V_i = \{v_i^0, v_i^1, v_i^2, \cdots, v_i^6\} \), \( 0 \leq i \leq 5 \). First decompose \( K_6^* \) into \( \{\overrightarrow{C}_3, \overrightarrow{T}_{21}\} \) as follows: Let

\[
\overrightarrow{H}_1 = (v_0v_1v_2), \\
\overrightarrow{H}_2 = (v_2v_4v_5), \\
\overrightarrow{H}_3 = (v_0v_4v_5) \quad \text{and} \\
\overrightarrow{T}_{21} = (v_1v_5v_1v_4v_3v_1v_0v_5v_2v_5v_3v_5v_4v_2v_3v_4v_5v_2).
\]

Clearly, \( \{\overrightarrow{H}_1, \overrightarrow{H}_2, \overrightarrow{H}_3, \overrightarrow{T}_{21}\} \) is a \( \{\overrightarrow{C}_3, \overrightarrow{T}_{21}\} \)-decomposition of \( K_6^* \), see Figure 3.5.

Now write \( K_6^* \times \overrightarrow{C}_7 = (3\overrightarrow{C}_3 \oplus \overrightarrow{T}_{21}) \times \overrightarrow{C}_7 = 3(\overrightarrow{C}_3 \times \overrightarrow{C}_7) \oplus (\overrightarrow{T}_{21} \times \overrightarrow{C}_7) = (\oplus \overrightarrow{C}_7) \oplus (\overrightarrow{T}_{21} \times \overrightarrow{C}_7) \), by Note 3.1. It remains to show that \( \overrightarrow{C}_{21} | \overrightarrow{T}_{21} \times \overrightarrow{C}_7 \).

Let \( \overrightarrow{H} = (v_0^0v_0^1v_0^2v_0^3v_0^5v_0^6v_0^1v_0^1v_3v_0^3v_0^1v_3v_0^3v_0^5v_0^6v_0^1v_2v_3v_1v_0^5v_2v_3v_1v_0^5v_2v_3v_1v_0^5) \) be the base cycle of \( \overrightarrow{T}_{21} \times \overrightarrow{C}_7 \) and let \( \rho \) be the permutation

\[
\rho = (V_0)(V_1)(V_2)(V_3)(V_4)(V_5).
\]

Then \( H, \rho(H), \rho^2(H), \cdots, \rho^6(H) \) is a \( \overrightarrow{C}_{21} \)-decomposition of \( \overrightarrow{T}_{21} \times \overrightarrow{C}_5 \). Thus \( \overrightarrow{C}_{21} | \overrightarrow{T}_{21} \times \overrightarrow{C}_5 \) and hence \( \overrightarrow{C}_{21} | \overrightarrow{C}_7 \times K_6^* \). □
Figure 3.5: $\{\overrightarrow{C}_3, \overrightarrow{T}_{21}\}$-decomposition of $K_6$. 
3.5  $\overrightarrow{C}_{3p}$-Decomposition of $(K_m \times K_n)^*$

**Theorem 3.5.1.** For $m, n \geq 3$, $\overrightarrow{C}_{3p} \mid (K_m \times K_n)^*$ if and only if

(i) $3p \mid m(m - 1)n(n - 1)$ and

(ii) $mn \geq 3p$.

**Proof.**

**Necessity:** Obvious necessary condition for the existence of a $\overrightarrow{C}_{3p}$-decomposition in $(K_m \times K_n)^*$ is the number of arcs of a $\overrightarrow{C}_{3p}$ must divide the number of arcs in $(K_m \times K_n)^*$, for $p = 5, 7$. Thus (i) holds. When $m = 2$ or $n = 2$, the graph becomes bipartite and hence no odd cycle exists, therefore $n, m \geq 3$. Another necessary condition is the number of vertices in $(K_m \times K_n)^*$ must be greater than or equal to number of vertices of $\overrightarrow{C}_{3p}$, for $p = 5, 7$. Thus (ii) holds.

**Sufficiency:** We deal the proof in two cases.

**Case 1:** Either $m$ or $n$ is odd.

Since the tensor product is commutative, without loss of generality assume that $m$ is odd. The remaining proof is given in different subcases.

**Case 1.1:** $\gcd (m(m - 1), 3p) = 3p$.

As $m$ is odd and $3p \mid m(m - 1)$, we have $\overrightarrow{C}_{3p} \mid K_m^*$, by Theorem 3.2.10. Therefore,

$$(K_m \times K_n)^* = (\oplus \overrightarrow{C}_{3p}) \times K_n^*$$

$$= \oplus (\overrightarrow{C}_{3p} \times K_n^*)$$
Case 1.2: gcd \((m(m-1), 3p) = p\).

By the necessary condition, \(p \mid m(m-1)\) implies \(3 \mid n(n-1)\).
Therefore, \(C_p \mid K_m^*\) and \(C_3 \mid K_n^*\), for \(n \neq 6\), by Theorem 3.2.10.

For \(n \neq 6\),

\[
(K_m \times K_n)^* = K_m^* \times K_n^* \\
= (\oplus C_p) \times (\oplus C_3) \\
= \oplus (C_p \times C_3) \\
= \oplus C_{3p}, \text{ by Note 3.1.}
\]

When \(n = 6\),

\[
(K_m \times K_n)^* = K_m^* \times K_6^* \\
= K_6^* \times K_m^* \\
= \oplus C_{3p}, \text{ by Lemmas 3.3.4 and 3.4.10.}
\]

Case 1.3: gcd\((m(m-1), 3p) = 3\).

By the necessary condition, \(3 \mid m(m-1)\) implies \(p \mid n(n-1)\). Therefore, \(C_3 \mid K_m^*\) and \(C_p \mid K_n^*\), by Theorem 3.2.10. Now

\[
(K_m \times K_n)^* = K_m^* \times K_n^* \\
= (\oplus C_3) \times (\oplus C_p) \\
= \oplus (C_3 \times C_p) \\
= \oplus C_{3p}, \text{ by Note 3.1.}
\]
Case 1.4: $\gcd(m(m - 1), 3p) = 1$.

By the necessary condition (i) we have $3p \mid n(n - 1)$. Therefore $\overrightarrow{C}_{3p} \mid K^*_n$, for $n \neq 6, 10$ when $p = 5$ and for $n \neq 7, 15$ when $p = 7$, by Theorem 3.2.10. Therefore

$$(K_m \times K_n)^* = K_m^* \times K_n^*$$

$$= (\oplus \overrightarrow{C}_{3p}) \times K_m^*$$

$$= \oplus (\overrightarrow{C}_{3p} \times K_m^*)$$

$$= \overrightarrow{C}_{3p}, \text{by Theorem 3.2.3.}$$

For $n = 6, 10$ when $p = 5$ the graphs are $(K_6 \times K_m)^*$, $(K_{10} \times K_m)^*$, and they are $\overrightarrow{C}_{15}$-decomposable, by Theorem 3.3.16. For $n \neq 7, 15$ when $p = 7$ the graphs are $(K_7 \times K_m)^*$ and $(K_{15} \times K_m)^*$ and are $\overrightarrow{C}_{15}$-decomposable by Theorem 3.4.9. Thus $\overrightarrow{C}_{3p} \mid (K_m \times K_n)^*$.

Case 2: Both $m$ and $n$ are even.

Here we have to prove that $\overrightarrow{C}_{3p} \mid (K_m \times K_n)^*$, $n \geq 4$ and for $p = 5, 7$.

Case 2.1: $\gcd(m(m - 1), 3p) = 3p$.

Here $3p \mid m(m - 1)$, therefore $\overrightarrow{C}_{3p} \mid K_m^*$, for $m \neq 6, 10$ when $p = 5$ and for $m \neq 7, 15$ when $p = 7$ by Theorem 3.2.10. Therefore

$$(K_m \times K_n)^* = (\oplus \overrightarrow{C}_{3p}) \times K_n^*$$

$$= \oplus (\overrightarrow{C}_{3p} \times K_n^*)$$

$$= \overrightarrow{C}_{3p}, \text{by Theorem 3.2.3.}$$

Thus $\overrightarrow{C}_{3p} \mid (K_m \times K_n)^*$. When $m = 6, 10, 7, 15$, we have $\overrightarrow{C}_{15} \mid (K_m \times K_n)^*$, for $m = 6, 10$ and $\overrightarrow{C}_{21} \mid (K_m \times K_n)^*$, for $n \geq 4$ and $m = 7, 15$, by Theorems 3.3.16 and 3.4.9, respectively.
Case 2.2: \( \gcd(m(m - 1), 15) = p. \)

By the necessary condition, \( p \mid m(m - 1) \) implies \( 3 \mid n(n - 1). \) Therefore \( \overrightarrow{C}_p \mid K_n^* \) and \( \overrightarrow{C}_3 \mid K_n^*, n \neq 6. \)

For \( n \neq 6, \)

\[
(K_m \times K_n)^* = K_m^* \times K_n^*
\]

\[
= (\oplus \overrightarrow{C}_p) \times (\oplus \overrightarrow{C}_3)
\]

\[
= \oplus (\overrightarrow{C}_p \times \overrightarrow{C}_3)
\]

\[
= \oplus \overrightarrow{C}_{3p}, \text{ by Theorem 3.2.10 and Note 3.1.}
\]

When \( n = 6, \)

\[
(K_m \times K_n)^* = K_m^* \times K_6^*
\]

\[
= K_6^* \times K_n^*
\]

\[
= \oplus \overrightarrow{C}_{3p}, \text{ by Lemmas 3.3.4 and 3.4.10.}
\]

Case 2.3: \( \gcd(m(m - 1), 3p) = 3. \)

Proof similar to Case 2.2 holds.

Case 2.4: \( \gcd(m(m - 1), 3p) = 1. \)

As \((K_m \times K_n)^* = (K_n \times K_m)^*\), the proof follows from Case 2.1. \( \square \)

**Theorem 3.5.2.** For \( m, n \geq 3, \overrightarrow{C}_{pq} \mid (K_m \times K_n)^* \) then

(i) \( pq \mid m(m - 1)n(n - 1) \) and (ii) \( mn \geq pq. \)

**Proof.** One of the obvious necessary conditions for the existence of \( \overrightarrow{C}_{pq} \) decomposition in \((K_m \times K_n)^*\) is \( pq \) must divide the number of edges in \((K_m \times K_n)^*\). Thus (i) holds. Another necessary condition is the number of vertices in \((K_m \times K_n)^*\) must be greater than or equal to number of vertices of \( \overrightarrow{C}_{pq}. \) Thus (ii) holds. \( \square \)
**Theorem 3.5.3.** For all primes $p, q \geq 3$ and $m, n \geq pq$, $\overrightarrow{C}_{pq} \mid (K_m \times K_n)^*$. 

**Proof.** We deal the proof in four cases.

**Case 1.** $\gcd(m(m - 1), pq) = pq$.

As $pq \mid m(m - 1)$, then $\overrightarrow{C}_{pq} \mid K_m^*$, by Theorem 3.2.10. Therefore

$$
(K_m \times K_n)^* = K_m^* \times K_n^*
$$

$$
= (\oplus \overrightarrow{C}_{pq}) \times K_n^*
$$

$$
= \oplus(\overrightarrow{C}_{pq} \times K_n^*)
$$

$$
= \overrightarrow{C}_{pq}, \text{by Theorem 3.2.3.}
$$

**Case 2.** $\gcd(m(m - 1), pq) = p$.

By the necessary condition, $p \mid m(m - 1)$ implies $q \mid n(n - 1)$. Therefore $\overrightarrow{C}_p \mid K_m^*$ and $\overrightarrow{C}_q \mid K_n^*$, by Theorem 3.2.10. We write

$$
(K_m \times K_n)^* = K_m^* \times K_n^*
$$

$$
= (\oplus \overrightarrow{C}_p) \times (\oplus \overrightarrow{C}_q)
$$

$$
= \oplus(\overrightarrow{C}_p \times \overrightarrow{C}_q)
$$

$$
= \oplus \overrightarrow{C}_{pq}, \text{by Note 3.1.}
$$

**Case 3.** $\gcd(m(m - 1), pq) = q$.

Similar proof of Case 2 holds.

**Case 4.** $\gcd(m(m - 1), pq) = 1$.

By condition (i) and the commutative property of tensor product, the proof follows from Case 1. Thus $\overrightarrow{C}_{pq} \mid (K_m \times K_n)^*$. □

**Note 3.2:** Our results answer Question 1 completely when $pq = 3p$, for $p = 5, 7$ and partially when $p, q \geq 3$ and $m, n \geq pq$. The question remains open for other values of $p, q$ and $m, n < pq$. □
3.6 \( \vec{C}_{3p} \)-Decomposition of \((K_m \circ \overline{K_n})^*\)

Some basic constructions in the form of Lemmas are presented in this section. Finally, the existence of \( \vec{C}_{3p} \)-decomposition of symmetric complete \( m \)-partite digraphs has been proved.

Lemma 3.6.1. For \( n \geq 3 \), \( \vec{C}_{15} \mid (K_6 \circ \overline{K_n})^* \)

Proof. Let \( \overrightarrow{H}_1 = (v_0v_1v_5v_1v_3v_4v_5v_2v_3v_4v_1v_0v_5) \) and \( \overrightarrow{H}_2 = (v_0v_2v_1v_2v_3v_2v_4v_3v_0v_4v_3v_0v_3v_2) \).

Clearly \( \{H_i, i = 1, 2\} \) is a \( \vec{T}_{15} \)-decomposition of \( K_6^* \). Note that \( \chi(H_i) = 3 \) and \( \Delta(H_i) = 8 \) for \( i = 1, 2 \). Hence by Theorem 3.2.11, we have \( \vec{C}_{15} \mid (H_i \circ \overline{K_n})^* \), for \( i = 1, 2 \) and \( n > 3 \). Therefore \( \vec{C}_{15} \mid (K_6 \circ \overline{K_n})^* \).

When \( n = 3 \),

\[
(K_6 \circ \overline{K_3})^* = K_6^* \circ \overline{K_3} = (\oplus \vec{C}_5) \circ \overline{K_3} = \oplus \vec{C}_{15}, \text{ by Theorems 3.2.5 and 3.2.10.} \]

Lemma 3.6.2. \( \vec{C}_{15} \mid (K_{10} \circ \overline{K_n})^* \)

Proof. Let \( \overrightarrow{H}_1 = (v_0v_8v_9v_0v_1v_2v_9v_2v_3v_2v_4v_3v_4v_5v_9) \), \( \overrightarrow{H}_2 = (v_0v_8v_0v_9v_2v_9v_3v_4v_3v_0v_4v_2v_1) \), \( \overrightarrow{H}_3 = (v_0v_8v_0v_3v_3v_9v_1v_2v_1v_9v_7v_7v_3v_0v_1v_3) \), \( \overrightarrow{H}_4 = (v_1v_8v_2v_5v_2v_8v_1v_7v_4v_7v_6v_7v_0v_3) \).
\[
\overrightarrow{H}_5 = (v_0 v_4 v_5 v_3 v_1 v_6 v_3 v_7 v_3 v_6 v_2 v_7 v_5 v_8 v_4), \\
\overrightarrow{H}_6 = (v_0 v_2 v_5 v_7 v_2 v_6 v_8 v_3 v_8 v_5 v_6 v_5 v_8 v_1 v_3).
\]

Clearly \(\{H_i, i = 1, 2, \ldots, 6\}\) is a \(\overrightarrow{T}_{15}\)-decomposition of \(K_{10}^*\). Note that \(\chi(H_i) = 3\) and \(\Delta(H_i) = 6\) for \(i = 1, 2, \ldots, 6\). Hence by Theorem 3.2.11, we have \(\overrightarrow{C}_{15} \mid (H_i \circ \overline{K}_n)^*\), for \(i = 1, 2, \ldots, 6\). Therefore \(\overrightarrow{C}_{15} \mid (K_{10} \circ \overline{K}_n)^*\). □

Lemma 3.6.3. For \(n \geq 3\), \(\overrightarrow{C}_{21} \mid (K_7 \circ \overline{K}_n)^*\)

**Proof.** Let
\[
\overrightarrow{H}_1 = (v_0 v_6 v_5 v_0 v_5 v_4 v_0 v_4 v_2 v_4 v_4 v_1 v_6 v_5 v_2 v_1 v_2 v_3 v_3) \quad \text{and} \\
\overrightarrow{H}_2 = (v_0 v_2 v_5 v_1 v_5 v_2 v_5 v_3 v_4 v_4 v_3 v_4 v_1 v_3 v_1 v_2 v_6 v_6 v_7 v_0 v_3 v_4).
\]

Clearly \(\{H_i, i = 1, 2\}\) is a \(\overrightarrow{T}_{21}\)-decomposition of \(K_7^*\). Note that \(\chi(H_i) = 3\) and \(\Delta(H_i) = 8\) for \(i = 1, 2\). Hence by Theorem 3.2.11, we have \(\overrightarrow{C}_{21} \mid (H_i \circ \overline{K}_n)^*\), for \(i = 1, 2\) and \(n > 3\). Therefore \(\overrightarrow{C}_{21} \mid (K_7 \circ \overline{K}_n)^*\).

When \(n = 3\),
\[
(K_7 \circ \overline{K}_3)^* = K_7^* \circ \overline{K}_3 \\
= (\oplus \overrightarrow{C}_7) \circ \overline{K}_3 \\
= \oplus \overrightarrow{C}_{21}, \quad \text{by Theorems 3.2.5 and 3.2.10, see Figure 3.6} \quad \Box
\]

Lemma 3.6.4. For \(n \geq 3\), \(\overrightarrow{C}_{21} \mid (K_{15} \circ \overline{K}_n)^*\)

**Proof.** Let
\[
\overrightarrow{H}_1 = (v_0 v_6 v_{12} v_5 v_{10} v_3 v_{13} v_8 v_{11} v_8 v_{12} v_6 v_{10} v_6 v_{11} v_6 v_0 v_7 v_0 v_3 v_4),
\]
\[ H_2 = (v_0 v_5 v_0 v_9 v_1 v_8 v_1 v_5 v_0 v_8 v_0 v_4 v_3 v_5 v_6 v_1 v_4 v_6 v_3 v_7 v_9), \]
\[ H_3 = (v_0 v_1 v_5 v_1 v_6 v_1 v_7 v_1 v_3 v_4 v_1 v_4 v_6 v_1 v_1 v_2 v_2 v_6 v_0 v_2 v_0 v_2 v_2), \]
\[ H_4 = (v_0 v_2 v_7 v_0 v_8 v_1 v_2 v_1 v_5 v_0 v_1 v_0 v_7 v_1 v_1 v_4 v_1 v_1 v_1 v_2 v_1 v_1 v_2 v_1), \]
\[ H_5 = (v_{10} v_{13} v_{11} v_{0} v_{11} v_{1} v_{11} v_{1} v_{3} v_{11} v_{13} v_{8} v_{2} v_{5} v_{2} v_{8} v_{13} v_{14} v_{8} v_{14} v_{12} v_{14}), \]
\[ H_6 = (v_{10} v_{9} v_{11} v_{9} v_{10} v_{11} v_{2} v_{1} v_{2} v_{2} v_{3} v_{1} v_{3} v_{2} v_{4} v_{1} v_{12} v_{0} v_{12} v_{2} v_{12}), \]
\[ H_7 = (v_{5} v_{8} v_{9} v_{12} v_{4} v_{12} v_{3} v_{12} v_{9} v_{13} v_{9} v_{14} v_{1} v_{14} v_{2} v_{10} v_{2} v_{14} v_{9}), \]
\[ H_8 = (v_{10} v_{1} v_{10} v_{1} v_{10} v_{1} v_{14} v_{0} v_{14} v_{3} v_{14} v_{4} v_{14} v_{4} v_{6} v_{4} v_{7} v_{8} v_{7} v_{4} v_{14} v_{13}), \]
\[ H_9 = (v_{10} v_{12} v_{7} v_{9} v_{4} v_{13} v_{8} v_{13} v_{2} v_{13} v_{1} v_{3} v_{13} v_{14} v_{9} v_{9} v_{9} v_{9} v_{9} v_{7} v_{12} v_{11}), \]
\[ H_{10} = (v_{5} v_{6} v_{13} v_{1} v_{13} v_{9} v_{7} v_{5} v_{7} v_{5} v_{9} v_{8} v_{11} v_{8} v_{4} v_{10} v_{4} v_{8} v_{5} v_{14}). \]

Clearly \( \{H_i, i = 1, 2, \ldots, 10\} \) is a \( \vec{T}_{21} \)-decomposition of \( K_{15}^* \), see Figure 3.7. Note that \( \chi(H_i) = 3 \) and \( \Delta(H_i) = 8 \) for \( i = 1, 2, \ldots, 10 \). Hence by Theorem 3.2.11, we have \( \vec{C}_{21} \mid (H_i \circ K_n)^* \), for \( i = 1, 2, \ldots, 10 \) and \( n > 3 \). Therefore \( \vec{C}_{21} \mid (K_{15} \circ K_n)^* \).

When \( n = 3 \),

\[
(K_{15} \circ K_3)^* = K_{15}^* \circ K_3
\]
\[
= (\oplus \vec{C}_7) \circ K_3
\]
\[
= \oplus \vec{C}_{21}, \text{ by Theorems 3.2.5 and 3.2.10.} \quad \Box
\]
Figure 3.6: $\overrightarrow{T}_{21}$-decomposition of $K_7^*$
Figure 3.7: $\overrightarrow{T}_{21}$-decomposition of $K_{15}^*$

(a) $\overrightarrow{H}_1$ of $K_{15}^*$

(b) $\overrightarrow{H}_2$ of $K_{15}^*$
Figure 3.7: $\overrightarrow{T}_{21}$-decomposition of $K_{15}^*$
Figure 3.7: $\overrightarrow{T}_{21}$-decomposition of $K_{15}^*$
Figure 3.7: $\overrightarrow{T}_{21}$-decomposition of $K_{15}$
Figure 3.7: $T_{21}$-decomposition of $K_{15}^*$
Lemma 3.6.5. For \( n \geq 3 \), \( \overrightarrow{C}_{33} | (K_{12} \circ K_n)^* \)

Proof. Let

\[
\overrightarrow{H}_1 = (v_0v_5v_3v_10v_16v_5v_17v_0v_0v_7v_0v_3v_0v_1v_0v_5v_3v_11v_0v_8v_9v_11, v_0v_5v_3v_2v_5v_1v_0v_1v_11v_1v_7v_1),
\]

\[
\overrightarrow{H}_2 = (v_0v_5v_4v_7v_9v_8v_3v_4v_8v_0v_0v_10v_0v_11v_7v_11, v_0v_1v_8v_1v_3v_2v_1v_5v_6v_5),
\]

\[
\overrightarrow{H}_3 = (v_0v_2v_4v_2v_10v_9v_1v_2v_3v_10v_3v_0v_1v_3v_3v_2v_7v_3v_4, v_1v_10v_1v_4v_3v_5v_7v_5v_11v_5v_3v_5v_4v_1v_4v_2),
\]

\[
\overrightarrow{H}_4 = (v_4v_9v_4v_10v_11v_8v_7v_6v_7v_8v_11v_10v_4v_11v_4v_3v_0v_3v_1v_3, v_2v_6v_2v_7v_2v_8v_5v_9v_5v_10v_5v_8v_2).
\]

Clearly \( \{H_i, i = 1, 2, \ldots, 4\} \) is a \( \overrightarrow{T}_{33} \)– decomposition of \( K_{12}^* \). Note that \( \chi(H_i) = 3 \) and \( \Delta(H_i) = 8 \) for \( i = 1, \ldots, 4 \). Hence by Theorem 3.2.11, we have \( \overrightarrow{C}_{33} | (H_i \circ K_n)^* \), for \( i = 1, \ldots, 4 \) and \( n > 3 \). Therefore \( \overrightarrow{C}_{33} | (K_{12} \circ K_n)^* \).

When \( n = 3 \),

\[
(K_{12} \circ K_3)^* = K_{12}^* \circ K_3
\]

\[
= (\oplus \overrightarrow{C}_{11}) \circ K_3
\]

\[
= \oplus \overrightarrow{C}_{33}, \text{ by Theorems 3.2.5 and 3.2.10.} \quad \Box
\]

Lemma 3.6.6. For \( n \geq 3 \), \( \overrightarrow{C}_{33} | (K_{22} \circ K_n)^* \)

Proof. Let

\[
\overrightarrow{H}_1 = (v_0v_5v_15v_5v_16v_5v_17v_5v_0v_0v_7v_0v_3v_0v_1v_0v_5v_3v_11v_0v_8v_9v_11, v_20v_1v_4v_1v_2v_5v_2v_8v_2v_20v_2v_3v_4),
\]
\[ \overrightarrow{H_2} = \{ v_0, v_7, v_9, v_0, v_20, v_21, v_20, v_4, v_3, v_13, v_2, v_2, v_15, v_3, v_2, v_7 \} , \]
\[ \overrightarrow{H_3} = \{ v_5, v_10, v_11, v_5, v_12, v_5, v_16, v_6, v_17, v_6, v_4, v_3, v_1, v_4, v_4, v_20, v_4, v_21, v_0 \} , \]
\[ \overrightarrow{H_4} = \{ v_5, v_13, v_14, v_5, v_21, v_5, v_0, v_15, v_9, v_16, v_0, v_8, v_16, v_3, v_15, v_7, v_8, v_7 \} , \]
\[ \overrightarrow{H_5} = \{ v_15, v_1, v_21, v_15, v_2, v_15, v_3, v_15, v_6, v_0, v_14, v_1, v_14, v_0, v_16, v_1, v_16, v_2, v_16 \} , \]
\[ \overrightarrow{H_6} = \{ v_15, v_1, v_15, v_4, v_14, v_2, v_21, v_2, v_14, v_3, v_4, v_15, v_20, v_15, v_19, v_0, v_18, v_17, v_17, v_3 \} , \]
\[ \overrightarrow{H_7} = \{ v_15, v_1, v_15, v_4, v_14, v_2, v_21, v_2, v_14, v_3, v_4, v_15, v_20, v_15, v_19, v_0, v_18, v_17, v_17, v_3 \} , \]
\[ \overrightarrow{H_8} = \{ v_15, v_1, v_15, v_4, v_14, v_2, v_21, v_2, v_14, v_3, v_4, v_15, v_20, v_15, v_19, v_0, v_18, v_17, v_17, v_3 \} , \]
\[ \overrightarrow{H_9} = \{ v_15, v_1, v_15, v_4, v_14, v_2, v_21, v_2, v_14, v_3, v_4, v_15, v_20, v_15, v_19, v_0, v_18, v_17, v_17, v_3 \} , \]
\[ \overrightarrow{H_10} = \{ v_15, v_1, v_15, v_4, v_14, v_2, v_21, v_2, v_14, v_3, v_4, v_15, v_20, v_15, v_19, v_0, v_18, v_17, v_17, v_3 \} , \]
\[ \overrightarrow{H_11} = \{ v_15, v_1, v_15, v_4, v_14, v_2, v_21, v_2, v_14, v_3, v_4, v_15, v_20, v_15, v_19, v_0, v_18, v_17, v_17, v_3 \} , \]
\[ \overrightarrow{H_12} = \{ v_15, v_1, v_15, v_4, v_14, v_2, v_21, v_2, v_14, v_3, v_4, v_15, v_20, v_15, v_19, v_0, v_18, v_17, v_17, v_3 \} , \]
Clearly \( \{H_i, i = 1, 2, \cdots, 14\} \) is a \( T_{33}^- \) decomposition of \( K_{22}^* \). Note that 
\( \chi(H_i) = 3 \) and \( \Delta(H_i) = 8 \) for \( i = 1, \cdots, 14 \). Hence by Theorem 3.2.11, we have \( \overrightarrow{C}_{33}|(H_i \circ K_n)^* \), for \( i = 1, \cdots, 14 \) and \( n > 3 \).
Therefore \( \overrightarrow{C}_{33}|(K_{22} \circ K_n)^* \).

When \( n = 3 \),

\[
(K_{22} \circ K_3)^* = K_{22}^* \circ K_3 \\
= (\oplus \overrightarrow{C}_{11}) \circ K_3 \\
= \oplus \overrightarrow{C}_{33}, \text{ by Theorems 3.2.5 and 3.2.10.} \quad \square
\]

**Lemma 3.6.7.** For \( n \geq 3 \), \( \overrightarrow{C}_{39}|(K_{13} \circ K_n)^* \)

**Proof.** Let

\[
\overrightarrow{H}_1 = (v_0v_3v_9v_3v_10v_4v_3v_14v_3v_11v_1v_2v_12v_1v_13v_1v_4v_1v_5v_1v_2v_1v_10v_4), \\
\overrightarrow{H}_2 = (v_0v_4v_8v_8v_3v_11v_1v_10v_7v_10v_9v_5v_5v_1v_3v_1v_11v_1), \\
\overrightarrow{H}_3 = (v_0v_1v_5v_12v_5v_12v_10v_1v_2v_12v_1v_4v_2v_4v_11v_1v_11v_1v_1v_3v_1v_5v_1v_10v_5v_11v_5),
\]
Clearly \( \{H_i, i = 1, 2, \ldots, 4\} \) is a \( T_{39} \)-decomposition of \( K_{13}^* \). Note that \( \chi(H_i) = 3 \) and \( \Delta(H_i) = 8 \) for \( i = 1, \ldots, 4 \). Hence by Theorem 3.2.11, we have \( \overrightarrow{C}_{39} | (H_i \circ K_n)^* \), for \( i = 1, \ldots, 4 \) and \( n > 3 \). Therefore \( \overrightarrow{C}_{39} | (K_{13} \circ K_n)^* \).

When \( n = 3 \),
\[
(K_{13} \circ K_3)^* = K_{13}^* \circ K_3 = (\oplus \overrightarrow{C}_{13}) \circ K_3
\]
\[
= \oplus \overrightarrow{C}_{39}, \text{ by Theorems 3.2.5 and 3.2.10.} \]

**Lemma 3.6.8.** For \( n \geq 3 \), \( \overrightarrow{C}_{39} | (K_{27} \circ K_n)^* \)

**Proof.** Let
\[
\overrightarrow{H}_1 = (v_0v_3v_10v_20v_23v_20v_10v_21v_12v_22v_14v_15v_16v_17v_18v_19v_20v_21v_22v_23v_24v_25v_26v_27v_28v_29v_30v_31v_32v_33v_34v_35v_36v_37v_38v_39),
\]
\[
\overrightarrow{H}_2 = (v_0v_3v_10v_3v_11v_12v_13v_14v_15v_16v_17v_18v_19v_20v_21v_22v_23v_24v_25v_26v_27v_28v_29v_30v_31v_32v_33v_34v_35v_36v_37v_38v_39),
\]
\[
\overrightarrow{H}_3 = (v_0v_3v_10v_3v_11v_12v_13v_14v_15v_16v_17v_18v_19v_20v_21v_22v_23v_24v_25v_26v_27v_28v_29v_30v_31v_32v_33v_34v_35v_36v_37v_38v_39),
\]
\[
\overrightarrow{H}_4 = (v_0v_3v_10v_3v_11v_12v_13v_14v_15v_16v_17v_18v_19v_20v_21v_22v_23v_24v_25v_26v_27v_28v_29v_30v_31v_32v_33v_34v_35v_36v_37v_38v_39),
\]
\[
\overrightarrow{H}_5 = (v_0v_3v_10v_3v_11v_12v_13v_14v_15v_16v_17v_18v_19v_20v_21v_22v_23v_24v_25v_26v_27v_28v_29v_30v_31v_32v_33v_34v_35v_36v_37v_38v_39),
\]
\[ \overrightarrow{H}_6 = (V_{10}V_{14}V_{25}V_{26}V_{25}V_{14}V_{13}V_{15}V_{13}V_{16}V_{19}V_{13}V_{12}V_{10}V_{12}V_{25}V_{13}V_{18}V_{23} \]
\[ V_{18}V_{24}V_{14}V_{24}V_{24}V_{14}V_{15}V_{13}V_{16}V_{10}V_{19}V_{18}V_{11}V_{18}V_{20}V_{20}V_{20}V_{16}V_{16}V_{16}V_{16}V_{22}V_{16}V_{17}V_{18} \]
\[ V_{19}V_{21}V_{19}V_{22}V_{12}V_{23}V_{12}V_{24}V_{26}V_{3}V_{13}V_{3}V_{26}V_{24}V_{12}V_{22}V_{19}V_{23}V_{19} \],
\[ \overrightarrow{H}_7 = (V_{15}V_{15}V_{17}V_{5}V_{15}V_{16}V_{20}V_{7}V_{20}V_{5}V_{20}V_{16}V_{14}V_{22}V_{16}V_{17}V_{18} \]
\[ V_{19}V_{21}V_{19}V_{22}V_{12}V_{23}V_{12}V_{24}V_{26}V_{3}V_{13}V_{3}V_{26}V_{24}V_{12}V_{22}V_{19}V_{23}V_{19} \],
\[ \overrightarrow{H}_8 = (V_{15}V_{19}V_{18}V_{16}V_{12}V_{26}V_{26}V_{6}V_{21}V_{13}V_{5}V_{21}V_{8}V_{21}V_{9}V_{21}V_{18}V_{17} \]
\[ V_{20}V_{10}V_{10}V_{11}V_{10}V_{14}V_{10}V_{20}V_{1}V_{20}V_{3}V_{20}V_{17}V_{22}V_{17}V_{17}V_{16} \],
\[ \overrightarrow{H}_9 = (V_{20}V_{24}V_{1}V_{24}V_{27}V_{20}V_{26}V_{20}V_{21}V_{22}V_{23}V_{24}V_{12}V_{12}V_{12}V_{10}V_{12}V_{9} \]
\[ V_{14}V_{9}V_{13}V_{9}V_{24}V_{3}V_{5}V_{3}V_{8}V_{3}V_{13}V_{6}V_{6}V_{3}V_{24}V_{4}V_{24} \],
\[ \overrightarrow{H}_{10} = (V_{20}V_{24}V_{1}V_{24}V_{27}V_{20}V_{26}V_{20}V_{21}V_{22}V_{23}V_{24}V_{12}V_{12}V_{12}V_{10}V_{12}V_{9} \]
\[ V_{14}V_{9}V_{13}V_{9}V_{24}V_{3}V_{5}V_{3}V_{8}V_{3}V_{13}V_{6}V_{6}V_{3}V_{24}V_{4}V_{24} \],
\[ \overrightarrow{H}_{11} = (V_{0}V_{2}V_{11}V_{25}V_{11}V_{5}V_{26}V_{6}V_{26}V_{5}V_{26}V_{21}V_{14}V_{20}V_{14}V_{23}V_{10}V_{24}V_{10}V_{23} \]
\[ V_{14}V_{24}V_{14}V_{21}V_{6}V_{21}V_{9}V_{21}V_{4}V_{4}V_{4}V_{4}V_{25}V_{4}V_{1}V_{3} \],
\[ \overrightarrow{H}_{12} = (V_{0}V_{3}V_{7}V_{3}V_{25}V_{3}V_{9}V_{11}V_{22}V_{11}V_{23}V_{17}V_{24}V_{17}V_{23}V_{11}V_{9}V_{13}V_{20}V_{13} \]
\[ V_{21}V_{13}V_{22}V_{13}V_{9}V_{14}V_{15}V_{14}V_{16}V_{14}V_{22}V_{14}V_{9}V_{3}V_{1}V_{4}V_{8}V_{4}V_{4} \],
\[ \overrightarrow{H}_{13} = (V_{5}V_{12}V_{5}V_{7}V_{10}V_{15}V_{5}V_{15}V_{5}V_{25}V_{15}V_{10}V_{16}V_{16}V_{16}V_{16}V_{16}V_{19} \]
\[ V_{5}V_{19}V_{4}V_{16}V_{10}V_{7}V_{11}V_{7}V_{14}V_{7}V_{9}V_{10}V_{12}V_{9}V_{25}V_{9}V_{9} \],
\[ \overrightarrow{H}_{14} = (V_{5}V_{5}V_{12}V_{5}V_{13}V_{5}V_{17}V_{21}V_{0}V_{21}V_{12}V_{6}V_{23}V_{1}V_{22}V_{3}V_{22}V_{4} \]
\[ V_{22}V_{1}V_{21}V_{2}V_{21}V_{17}V_{8}V_{6}V_{16}V_{23}V_{16}V_{16}V_{16}V_{16}V_{12}V_{12}V_{12}V_{13}V_{16}V_{16}V_{7} \],
\[ \overrightarrow{H}_{15} = (V_{10}V_{12}V_{14}V_{17}V_{4}V_{17}V_{14}V_{18}V_{18}V_{14}V_{19}V_{20}V_{19}V_{24}V_{19}V_{26}V_{19}V_{14}V_{11} \]
\[ V_{13}V_{17}V_{13}V_{18}V_{13}V_{23}V_{2}V_{22}V_{0}V_{25}V_{0}V_{22}V_{23}V_{25}V_{23}V_{23}V_{23}V_{23}V_{13} \],
\[ \overrightarrow{H}_{16} = (V_{10}V_{13}V_{25}V_{15}V_{2}V_{15}V_{5}V_{15}V_{4}V_{10}V_{4}V_{11}V_{14}V_{12}V_{4}V_{13}V_{15}V_{25}V_{13}V_{24}V_{7} \]
\[ V_{17}V_{7}V_{24}V_{16}V_{26}V_{26}V_{26}V_{24}V_{13}V_{11}V_{14}V_{26}V_{8}V_{18}V_{8}V_{26}V_{14}V_{12} \].
Clearly \( \{ H_i, i = 1, 2, \ldots, 18 \} \) is a \( T_{39} \)-decomposition of \( K_{13} \). Note that \( \chi(H_i) = 3 \) and \( \Delta(H_i) = 8 \) for \( i = 1, \ldots, 18 \). Hence by Theorem 3.2.11, we have \( C_{39} | (H_i \circ K_n)^* \), for \( i = 1, \ldots, 18 \) and \( n > 3 \).

Therefore \( C_{39} \| (K_{27} \circ K_3)^* \).

When \( n = 3 \),

\[
(K_{27} \circ K_3)^* = K_{27}^* \circ K_3
= (\oplus C_{13}) \circ K_3
= \oplus C_{39}, \quad \text{by Theorems 3.2.5 and 3.2.10.} \]

Now the main result of this section is given below.

**Theorem 3.6.9.** For \( p = 5, 7, 11, 13 \), \( C_{3p} \| (K_m \circ K_n)^* \) if and only if

(i) \( 3p | m(m - 1)n^2 \),

(ii) \( m \geq 3, \; mn \geq 3p \).

**Proof.**

**Necessity:** The condition (i) can be obtained by counting the number of edges of \( (K_m \circ K_n)^* \). When \( m = 2 \), the graph is bipartite and hence no odd cycle exists. Therefore \( m \geq 3 \) and \( mn \geq 3p \), as the number vertices in the graph must be greater than or equal to number of vertices of \( C_{3p} \). Thus (ii)
holds.

**Sufficiency:**

**Case 1:** $m = \text{odd.}$

There are four sub cases.

**Sub case 1.1:** $\gcd(m(m-1), 3p) = 3p$, then $3p \mid m(m-1)$. If $m \geq 3p$, by Theorem 3.2.10, $\overrightarrow{C_{3p}} \mid K_m^*$. Therefore

\[
(K_m \circ K_n)^* = K_m^* \circ K_n \\
= (\oplus \overrightarrow{C_{3p}}) \circ K_n \\
= \oplus \left( \overrightarrow{C_{3p}} \circ K_n \right) \\
= (\oplus \overrightarrow{C_{3p}}), \text{ by Corollary 3.2.8.}
\]

If $m < 3p$ and $3p \mid m(m-1)$ then $m = p, 2p + 1$, for $p = 7, 13$. Therefore, it is enough to show that $\overrightarrow{C_{3p}} \mid (K_p \circ K_n)^*$ and $\overrightarrow{C_{3p}} \mid (K_{2p+1} \circ K_n)^*$. By Lemmas 3.6.3, 3.6.4, 3.6.7 and 3.6.8, we have

\[
\overrightarrow{C_{3p}} \mid (K_p \circ K_n)^* \text{ and } \overrightarrow{C_{3p}} \mid (K_{2p+1} \circ K_n)^*, \text{ when } p = 7, 13.
\]

**Sub case 1.2:** $\gcd(m(m-1), 3p) = p$. Then $p \mid m(m-1)$ and $3p \mid m(m-1)n^2$ implies $3 \mid n$, that is $n = 3r$. By Theorem 3.2.10, $\overrightarrow{C_p} \mid K_m^*$. Therefore, we get

\[
(K_m \circ K_n)^* = K_m^* \circ K_n \\
= (\oplus \overrightarrow{C_p}) \circ K_{3r} \\
= \oplus \left( \overrightarrow{C_p} \circ K_{3r} \right) \\
= (\oplus (\overrightarrow{C_p} \circ K_3)) \circ K_r \\
= (\oplus \overrightarrow{C_{3p}}) \circ K_r = (\oplus (\overrightarrow{C_{3p}} \circ K_r)) \\
= (\oplus \overrightarrow{C_{3p}}), \text{ by Corollary 3.2.8 and Theorem 3.2.5.}
Hence $C_{3p} | (K_m \circ K_n)^*$. 

**Sub case 1.3:** $\gcd(m(m-1), 3p) = 3$. Then $3 | m(m-1)$ and $3p | m(m-1)n^2$ implies $p | n$, that is $n = pr$. By Theorem 3.2.10, $C_{3} | K_m^*$. Therefore,

$$(K_m \circ K_n)^* = K_m^* \circ K_n$$

$$= (\oplus C_3) \circ K_{pr}$$

$$= \oplus (C_3 \circ K_p) \circ K_r$$

$$= (\oplus C_{3p}) \circ K_r$$

$$= \oplus (C_{3p} \circ K_r) = \oplus C_{3p}, \text{ by Corollary 3.2.8 and Theorem 3.2.5.}$$

Hence $C_{3p} | (K_m \circ K_n)^*$. 

**Sub case 1.4:** $\gcd(m(m-1), 3p) = 1$, $3p | n^2$ implies $3p | n$. That is $n = 3pr$.

Let $m = 2s + 1$. Therefore,

$$(K_m \circ K_n) = K_{2s+1} \circ K_{3pr}$$

$$= (K_{2s+1} \circ K_p) \circ K_{3r}$$

$$= \oplus (C_p) \circ K_{3r}$$

$$= \oplus (C_p \circ K_3) \circ K_r$$

$$= \oplus (C_{3p} \circ K_r)$$

$$= \oplus C_{3p}, \text{ by using Corollary 3.2.8 and Theorems 3.2.10 and 3.2.12.}$$

Thus we get $C_{3p} | (K_m \circ K_n)^*$. 

**Case 2:** $m = \text{even}$. There are four sub cases.

**Sub case 2.1:** $\gcd(m(m-1), 3p) = 3p$, then $3p | m(m-1)$
If $m \geq 3p$, by Theorem 3.2.10, $\overrightarrow{C}_{3p} | K_m^\ast$. Therefore

$$
(K_m \circ \overrightarrow{K_n})^\ast = K_m^\ast \circ \overrightarrow{K_n} = (\oplus \overrightarrow{C}_{3p}) \circ \overrightarrow{K_n} = \oplus \left( \overrightarrow{C}_{3p} \circ \overrightarrow{K_n} \right).
$$

Now $\overrightarrow{C}_{3p} \mid \left( \overrightarrow{C}_{3p} \circ \overrightarrow{K_n} \right)$ by Corollary 3.2.8 and hence $\overrightarrow{C}_{3p} \mid (K_m \circ K_n)^\ast$.

If $m < 3p$, and $3 \mid m(m - 1)$ implies $m = p + 1, 2p$, for $p = 5, 11$. Therefore, it is enough to show that $\overrightarrow{C}_{3p} \mid (K_{p+1} \circ \overrightarrow{K_n})^\ast$, and $\overrightarrow{C}_{3p} \mid (K_{2p} \circ \overrightarrow{K_n})^\ast$. By Lemmas 3.6.1, 3.6.2, 3.6.5 and 3.6.6, we have $\overrightarrow{C}_{3p} \mid (K_{p+1} \circ \overrightarrow{K_n})^\ast$ and $\overrightarrow{C}_{3p} \mid (K_{2p} \circ \overrightarrow{K_n})^\ast$, when $p = 5, 11$.

**Sub case 2.2**: $gcd(m(m - 1), 3p) = p$. Then by (i), $p \mid m(m - 1)$ implies $m \geq p$ and $3 \mid n$. That is $n = 3r$. By Theorem 3.2.10, $\overrightarrow{C}_p \mid K_m^\ast$. Therefore,

$$
(K_m \circ \overrightarrow{K_n})^\ast = K_m^\ast \circ \overrightarrow{K_n} = (\oplus \overrightarrow{C}_p) \circ \overrightarrow{K_{3r}} = \oplus \left( \overrightarrow{C}_p \circ \overrightarrow{K_{3r}} \right).
$$

Hence $\overrightarrow{C}_{3p} \mid (K_m \circ \overrightarrow{K_n})^\ast$.

**Sub case 2.3**: $gcd(m(m - 1), 3p) = 3$. Then by (i), $3 \mid m(m - 1)$ implies $p \mid n$ that is, $n = pr$. By Theorem 3.2.10, $\overrightarrow{C}_3 \mid K_m^\ast$, when $m \neq 6$. Therefore,

$$
(K_m \circ \overrightarrow{K_n})^\ast = K_m^\ast \circ \overrightarrow{K_n} = (\oplus \overrightarrow{C}_3) \circ \overrightarrow{K_{pr}}.
$$
Hence $\overline{C}_{3p} | (\overline{C}_3 \circ \overline{K}_{3p})$. When $m = 6$, by the hypothesis $p = 5$. By Lemma 3.6.1, $\overline{C}_{15} | (K_6 \circ \overline{K}_{5r})^*$. Hence $\overline{C}_{3p} | (K_m \circ \overline{K}_n)^*$.

**Sub case 2.4:** $\gcd(m(m - 1), 3p) = 1$. By Theorem 3.2.13, we have $\overline{C}_{3p} | (K_m \circ \overline{K}_n)^*$. □

We raise the following.

**Question:** Does there exist $T_{pq}$-decompositions in $K_m^*$, $m < pq$ with $\chi(T_{pq}) \leq 3$ and $\Delta(T_{pq}) \leq 6$?

**Remark 3.6.10.** If the above question is true, then by Theorem 3.2.11, we have $\overline{C}_{pq} | K_m^* \circ \overline{K}_t$. □