Gregarious Cycle
Factorizations of
Product Graphs
Chapter 2

Gregarious Cycle Factorizations of Product Graphs

2.1 Introduction

In [35], Hell and Rosa have introduced the concept of $G$-designs. Many types of $G$-designs can be found in [16, 34, 70]. When $G$ is a digraph on $k$ vertices, a resolvable $(m, n, k, \lambda)$ multipartite $G$-design is a factorization of $K_{m,n}^*(\lambda)$ into $G$-factors [70]. So, resolvable $(k, n, km, \lambda)$ multipartite $\hat{C}_{km}$-design is nothing but a $\hat{C}_{km}$-factorization of $K_{k,n}^*(\lambda)$, where $m|n$. Resolvable $(m, n, k, \lambda)$ multipartite $K_k$-design is a particular type of partially balanced incomplete block design ($PBIBD$).

$\hat{C}_k$-factorization of complete bipartite and tripartite symmetric digraphs were completely solved by Ushio [70–72]. $\hat{C}_k$-factorization of complete $k$-partite (for odd $k$) symmetric digraphs was completely solved by Muthusamy [57] and the same is left open for even $k$. In Section 2.3, it is shown that
$K_{2m}(\lambda) \circ K_{2r}$ and $K_{2m}(2\lambda) \circ K_{2s+1}$ are $\hat{C}_{2m}$-factorable. As a consequence, a resolvable $(k, n, k, 2\lambda)$ multipartite $\hat{C}_k$-design exists for even $k$. These results together with the results in [57] completely settle the existence of a resolvable $(k, n, k, 2\lambda)$ multipartite $\hat{C}_k$-design for all $k \geq 3$, $n \geq 1$ and $\lambda \geq 1$. In Section 2.4, the $\hat{C}_k$-factorizations of the regular $k$-partite symmetric digraphs are considered and it has been proved that $(K_m \times K_n)^*$ is $\hat{C}_m$-factorable for all $n > 2$, $m \geq 3$. Consequently, this proves the existence of a $\hat{C}_m$-factorization of $(K_m \circ K_n)^*$ when $m$ is odd.

2.2 Preliminary Results

Notation: Let $V(K_m) = V(K_n^*) = \{v_0, v_1, \ldots, v_{m-1}\}$ and

$$V(K_n) = \{u_0, u_1, \ldots, u_{n-1}\}.$$

As the following result extensively used in the construction of $\hat{C}_{2m}$-factorization of $K_{2m}(\lambda) \circ K_{2r}$ and $K_{2m}(2\lambda) \circ K_{2s+1}$, the proof of Walecki's construction is given here.

Theorem 2.2.1. [2] The complete graph $K_n$ has a Hamilton decomposition for all $n$.

Proof. The result is trivially true for $n = 1$ and $n = 2$. Let $n = 2m + 1 \geq 3$ be odd. Let the vertices of $K_n$ be labeled $v_0, v_1, v_2, \ldots, v_{2m}$. Let $C$ be the Hamilton cycle $v_0v_1v_2v_{2m}v_3v_{2m-1}v_4v_{2m-2} \ldots v_mv_{m+1}v_0$ and let $\sigma$ be the permutation $(v_0)(v_1v_2v_3v_{2m-1}v_{2m})$. Then $C, \sigma(C), \sigma^2(C), \ldots, \sigma^{m-1}(C)$ is a Hamilton decomposition of $K_n$. When $n = 2m \geq 4$ is even, let the vertices of $K_n$ be labeled $v_0, v_1, v_2, \ldots, v_{2m-1}$. Let $C$ be
the Hamilton cycle \( v_0 v_1 v_2 v_{2m-1} v_3 v_{2m-2} \ldots v_{m-1} v_m v_{m+1} v_0 \) and \( \sigma \) be the permutation \((v_0)(v_1 v_2 v_3 \ldots v_{2m-2} v_{2m-1})\). Then \( C, \sigma(C), \sigma^2(C), \ldots, \sigma^{m-2}(C) \) are \( m-1 \) edge disjoint Hamilton cycles. The remaining edges \( v_0 v_m, v_{m-1} v_{m+1}, v_{m-2} v_{m+2}, \ldots, v_1 v_{2m-1} \) form a perfect matching. \( \square \)

**Remark 2.2.2.** One can easily check that the union of the last Hamilton cycle and the 1-factor of \( K_{2m} \) from Walecki’s construction is a cubic graph

\[
G = F_1 \oplus F_2 \oplus F_3, \quad \text{where}
\]

\[
F_1 = \bigcup_{i=0}^{m-1} \{v_{2i}v_{2i+1}\},
\]

\[
F_2 = \bigcup_{i=0}^{m-1} \{v_{2i+1}v_{2i+2}\} \quad \text{and}
\]

\[
F_3 = \bigcup_{i=0}^{m-3} \{v_{2i+1}v_{2i+4}\} \oplus \{v_0v_2\} \oplus \{v_{2m-3}v_{2m-1}\},
\]

also it is seen that \( \{F_1, F_2, F_3\} \) are perfect 1-factors of \( G \). \( \square \)

The following are the results, which are used in the proof of our results.

**Lemma 2.2.3.** [59] If \( n \mid m \) and \( m/n = p \), then \( C_{2kn} \parallel C_{2k} \circ \overline{K}_m \) for all integers \( k, m \geq 2 \) and \( n \geq 1 \). \( \square \)

**Theorem 2.2.4.** [57] If \( m \mid n \) and \( n \neq 2 \) (mod 4) when \( k \) is odd, then \( C_{km} \parallel \overline{C}_k \circ \overline{K}_n \). \( \square \)

**Theorem 2.2.5.** [59] If \( n \mid m \) and \( m \neq 2 \) (mod 4) when \( k \) is odd, then \( C_{kn} \parallel C_k \times K_m \). \( \square \)

**Lemma 2.2.6.** [49] There exists a pair of mutually orthogonal latin squares of order \( n \) for every \( n \neq 2, 6 \). \( \square \)

**Theorem 2.2.7.** [63] For \( k \geq 3 \), \( C_k \parallel C_k \circ \overline{K}_n \) if and only if \((k,n) \neq \{(2m+1,2), (3,6)\}\). \( \square \)
2.3 Gregarious Cycle Factorizations of Wreath Product of Graphs

In this section, the existence of $\hat{C}_{2m}$-factorizations of complete multipartite symmetric digraphs are considered and some basic constructions which are required to prove the main result are also given.

Lemma 2.3.1. $\hat{C}_{2m} \parallel G \circ K_2$, where $G = F_1 \oplus F_2 \oplus F_3$, where

\[
F_1 = \bigcup_{i=0}^{m-1} \{v_{2i}v_{2i+1}\}, \\
F_2 = \bigcup_{i=0}^{m-1} \{v_{2i+1}v_{2i+2}\}, \text{ and} \\
F_3 = \bigcup_{i=0}^{m-3} \{v_{2i+1}v_{2i+4}\} \oplus \{v_0v_2\} \oplus \{v_{2m-3}v_{2m-1}\} \text{ and}
\]

$|G| = 2m \equiv 0 \pmod{4}$.

Proof. Let $V(G) = \{v_0, v_1, \ldots, v_{2m-1}\}$. Then $V(G \circ K_2) = \bigcup_{i=0}^{2m-1} V_i$, where $V_i = \{v^i_0, v^i_1\}$. Now $\hat{C}_{2m}$-factors of $G \circ K_2$ are constructed as follows: Let

\[
G_1 = \bigcup_{i=0}^{m-1} \alpha_0(V_{2i}, V_{2i+1}) \oplus \bigcup_{i=0}^{m-3} \alpha_0(V_{2i+1}, V_{2i+4}) \\
\quad \oplus \alpha_0(V_0, V_2) \oplus \alpha_0(V_{2m-3}, V_{2m-1}), \\
G_2 = \bigcup_{i=0}^{m-1} \alpha_0(V_{2i+1}, V_{2i+2}) \oplus \bigcup_{i=0}^{m-3} \alpha_1(V_{2i+1}, V_{2i+4}) \oplus \alpha_1(V_0, V_2) \\
\quad \oplus \alpha_1(V_{2m-3}, V_{2m-1})
\]

and

\[
G_3 = \bigcup_{i=0}^{2m-1} \alpha_1(V_i, V_{i+1}),
\]

where the subscripts are taken modulo $2m - 1$ with residues $0, 1, \ldots, 2m - 1$.

Obviously, $G_1$, $G_2$, and $G_3$ are the $\hat{C}_{2m}$-factors of $G \circ K_2$, see Figure 2.1.

Thus $\hat{C}_{2m} \parallel G \circ K_2$. □
Lemma 2.3.2. $\tilde{C}_{2m} \parallel K_{2m} \circ \overline{K}_2$, where $2m \equiv 0 \pmod{4}$.

**Proof.** By Walecki's construction, we have $K_{2m} = H_1 \oplus H_2 \oplus \cdots \oplus H_{m-1} \oplus F$ where $H_i$'s are the Hamilton cycles and $F$ is the 1-factor of $K_{2m}$. Therefore
\[ K_{2m} \circ \overline{K}_2 = (H_1 \oplus H_2 \oplus \cdots \oplus H_{m-1} \oplus F) \circ \overline{K}_2 = (H_1 \circ \overline{K}_2) \oplus \cdots \oplus (H_{m-2} \circ \overline{K}_2) \oplus (H_{m-1} \oplus F) \circ \overline{K}_2. \]

It is known by Lemma 2.2.3, \( \hat{C}_{2m} \parallel H_1 \circ \overline{K}_2 \) and by Lemma 2.3.1, \( \hat{C}_{2m} \parallel (H_{m-1} \oplus F) \circ \overline{K}_2 \), since \( H_{m-1} \oplus F \) is isomorphic to \( G \) considered in Lemma 2.3.1. Hence \( \hat{C}_{2m} \parallel K_{2m} \circ \overline{K}_2 \).

**Corollary 2.3.3.** \( \hat{C}_{2m} \parallel K_{2m}^* \circ \overline{K}_2 \) when \( 2m \equiv 0 \pmod{4} \).

**Theorem 2.3.4.** \( \hat{C}_{2m} \parallel K_{2m} \circ \overline{K}_{2r} \) when \( 2m \equiv 0 \pmod{4} \).

**Proof.** Now construct a new graph by considering disjoint subsets of size \( r \) (which can be obtained by dividing each part of \( K_{2m} \circ \overline{K}_{2r} \) into subsets of size \( r \)) as its vertex set and two of them are adjacent if the corresponding subsets form a \( K_{r,r} \) in \( K_{2m} \circ \overline{K}_{2r} \). The new graph is isomorphic to \( K_{2m} \circ \overline{K}_2 \). Then by Lemma 2.3.2, we have \( \hat{C}_{2m} \parallel K_{2m} \circ \overline{K}_2 \). Now by blowing up the vertices of \( K_{2m} \circ \overline{K}_2 \) into \( r \) vertices, each \( \hat{C}_{2m} \)-factor of \( K_{2m} \circ \overline{K}_2 \) becomes a graph isomorphic to \( 2(C_{2m} \circ \overline{K}_r) \). That is, \( K_{2m} \circ \overline{K}_{2r} = \oplus 2(C_{2m} \circ \overline{K}_r) \). Again by Lemma 2.2.3, \( \hat{C}_{2m} \parallel C_{2m} \circ \overline{K}_r \) and hence \( \hat{C}_{2m} \parallel K_{2m} \circ \overline{K}_{2r} \).

**Corollary 2.3.5.** \( \hat{C}_{2m} \parallel K_{2m}^* \circ \overline{K}_{2r} \) when \( 2m \equiv 0 \pmod{4} \).

**Corollary 2.3.6.** \( \hat{C}_{2m} \parallel K_{2m}^*(\lambda) \circ \overline{K}_{2r} \) when \( 2m \equiv 0 \pmod{4} \).

**Lemma 2.3.7.** \( \hat{C}_{2m} \parallel G^* \circ \overline{K}_2 \), where \( G \) is the graph considered in Remark 2.2.2, with \( |G| = 2m \equiv 2 \pmod{4} \).
Proof. Let $V(G^* \circ \mathcal{K}_2) = \bigcup_{i=0}^{2m-1} V_i$, where $V_i = \{v_i^0, v_i^1\}$. Now six $\hat{C}_{2m}$-factors of $G^* \circ \mathcal{K}_2$ are constructed as follows: Let

\begin{align*}
G_1 &= \{(v_0^0v_2^0v_3^0v_6^0v_7^0\ldots v_{2m-4}^0v_{2m-3}^0v_{2m-1}^0v_{2m-2}^0v_{2m-5}^0v_{2m-6}^0\ldots v_5^0v_4^0v_3^0)\}, \\
&\quad \{(v_0^1v_1^1v_2^1v_3^1\ldots v_{2m-6}^1v_{2m-5}^1v_{2m-2}^1v_{2m-1}^1v_{2m-3}^1v_{2m-4}^1\ldots v_1^1v_0^1v_3^1v_2^1)\}, \\
G_2 &= \{(v_0^0v_1^0v_4^0v_5^0\ldots v_{2m-6}^0v_{2m-5}^0v_{2m-2}^0v_{2m-1}^0v_{2m-3}^0v_{2m-4}^0\ldots v_6^0v_3^0v_2^0)\}, \\
&\quad \{(v_0^1v_2^1v_3^1v_6^1\ldots v_{2m-4}^1v_{2m-3}^1v_{2m-5}^1v_{2m-2}^1v_{2m-1}^1v_{2m-3}^1v_{2m-4}^1\ldots v_2^1v_3^1v_6^1v_2^1)\}, \\
G_3 &= \{(v_0^0v_2^0v_4^0v_5^0v_6^0\ldots v_{2m-4}^0v_{2m-5}^0v_{2m-2}^0v_{2m-3}^0v_{2m-1}^0v_{2m-2}^0v_{2m-3}^0v_{2m-1}^0)\}, \\
&\quad \{(v_0^1v_2^1v_3^1v_6^1v_7^1\ldots v_{2m-4}^1v_{2m-3}^1v_{2m-5}^1v_{2m-2}^1v_{2m-1}^1v_{2m-3}^1v_{2m-4}^1\ldots v_1^1v_2^1v_3^1v_6^1v_7^1)\}, \\
G_4 &= \{(v_0^0v_{2m-1}^0v_{2m-2}^0v_{2m-3}^0v_{2m-4}^0v_{2m-5}^0v_{2m-6}^0v_{2m-7}^0\ldots v_5^0v_6^0v_3^0v_4^0v_1^0v_2^0)\}, \\
&\quad \{(v_0^1v_{2m-1}^1v_{2m-2}^1v_{2m-3}^1v_{2m-4}^1v_{2m-5}^1v_{2m-6}^1v_{2m-7}^1\ldots v_5^1v_6^1v_3^1v_4^1v_1^1v_2^1)\}, \\
G_5 &= \{(v_0^0v_{2m-1}^0v_2^0v_3^0v_6^0v_7^0\ldots v_{2m-4}^0v_{2m-3}^0v_{2m-2}^0v_{2m-1}^0v_{2m-2}^0v_{2m-3}^0v_{2m-1}^0)\}, \\
&\quad \{(v_0^1v_{2m-1}^1v_{2m-2}^1v_{2m-3}^1v_{2m-4}^1v_{2m-5}^1v_{2m-6}^1v_{2m-7}^1\ldots v_5^1v_6^1v_3^1v_4^1v_1^1v_2^1)\}, \\
G_6 &= \{(v_0^0v_{2m-1}^0v_{2m-2}^0v_{2m-3}^0v_{2m-4}^0v_{2m-5}^0v_{2m-6}^0v_{2m-7}^0\ldots v_5^0v_6^0v_3^0v_4^0v_1^0v_2^0)\}, \\
&\quad \{(v_0^1v_{2m-1}^1v_{2m-2}^1v_{2m-3}^1v_{2m-4}^1v_{2m-5}^1v_{2m-6}^1v_{2m-7}^1\ldots v_5^1v_6^1v_3^1v_4^1v_1^1v_2^1)\}.
\end{align*}

The structure of $G$ in the hypothesis makes clear that each $G_i$, $1 \leq i \leq 6$, is a $\hat{C}_{2m}$-factor of $G^* \circ \mathcal{K}_2$, see Figure 2.2. Hence $\hat{C}_{2m} \parallel G^* \circ \mathcal{K}_2$. □
Figure 2.2: $\{G_1, G_2\}$ of $G^* \circ K_2$. 
\{G_3, G_4, G_5, G_6\} of $G^* \circ \overline{K}_2$.

Figure 2.2: $\hat{C}_{10} \parallel G^* \circ \overline{K}_2$, when $|G| = 10$. 
Theorem 2.3.8. \( \hat{C}_{2m} \parallel K^*_{2m} \circ \overline{K}_2 \) when \( 2m \equiv 2 \text{ (mod 4)} \).

**Proof.** By Walecki's construction we have \( K_{2m} = H_1 \oplus H_2 \oplus \cdots \oplus H_{m-1} \oplus F \), where \( H_i \)'s are the Hamilton cycles and \( F \) is the 1-factor of \( K_{2m} \). Therefore

\[
K^*_{2m} \circ \overline{K}_2 = (H_1 \oplus H_2 \oplus \cdots \oplus H_{m-1} \oplus F)^* \circ \overline{K}_2
\]

\[
= (H_1^* \circ \overline{K}_2) \oplus \cdots \oplus (H_{m-2}^* \circ \overline{K}_2) \oplus (H_{m-1} \oplus F)^* \circ \overline{K}_2.
\]

By Lemma 2.2.3, \( \hat{C}_{2m} \parallel (H_i^* \circ \overline{K}_2) \) for \( i = 1, 2, \ldots, m-2 \), and by Lemma 2.3.7, \( \hat{C}_{2m} \parallel (H_{m-1} \oplus F)^* \circ \overline{K}_2 \). Thus \( \hat{C}_{2m} \parallel K^*_{2m} \circ \overline{K}_2 \). \( \square \)

Theorem 2.3.9. \( \hat{C}_{2m} \parallel K^*_{2m} \circ \overline{K}_{2r}, 2m \equiv 2 \text{ (mod 4)} \).

**Proof.** Now construct a new graph by considering the subsets of size \( r \) (which can be obtained by dividing each part of \( K^*_{2m} \circ \overline{K}_{2r} \) into subsets of size \( r \)) as its vertex set and two of them are adjacent if the corresponding subsets form a \( K^*_{r,r} \) in \( K^*_{2m} \circ \overline{K}_{2r} \). The new graph is isomorphic to \( K^*_{2m} \circ \overline{K}_2 \).

Then by Theorem 2.3.8, we have \( \hat{C}_{2m} \parallel K^*_{2m} \circ \overline{K}_2 \). Now blowing up the vertices of \( K^*_{2m} \circ \overline{K}_2 \), into \( r \) vertices, each \( \hat{C}_{2m} \)-factor of \( K^*_{2m} \circ \overline{K}_2 \) becomes a graph isomorphic to \( 2(\overrightarrow{C}_{2m} \circ \overline{K}_r) \). That is, \( K^*_{2m} \circ \overline{K}_{2r} = \oplus 2(\overrightarrow{C}_{2m} \circ \overline{K}_r) \).

Again by Lemma 2.2.3, \( \hat{C}_{2m} \parallel \overrightarrow{C}_{2m} \circ \overline{K}_r \) and hence \( \hat{C}_{2m} \parallel K^*_{2m} \circ \overline{K}_{2r} \). \( \square \)

Corollary 2.3.10. \( \hat{C}_{2m} \parallel K^*_{2m}(\lambda) \circ \overline{K}_{2r}, 2m \equiv 2 \text{ (mod 4)} \).

**Proof.** Proof of this theorem follows from Corollaries 2.3.6 and 2.3.10. \( \square \)
Lemma 2.3.12. \( \hat{C}_{2m} \parallel G(2) \circ K_{2s+1} \), where \( G \) is the graph given in Remark 2.2.2.

Proof. Let \( v_0, v_1, \ldots, v_{2m-1} \) be the vertices of \( G \) and \( u_0, u_2, \ldots, u_{2s} \) be the vertices of \( K_{2s+1} \). Let \( V_i = \{ v_i^0, v_i^1, \ldots, v_i^{2s} \} \), \( 0 \leq i \leq 2m - 1 \) be the set of vertices of \( G \circ K_{2s+1} \) correspond to the vertex \( v_i \) of \( G \). Now

\[
G(2) \circ K_{2s+1} = \{ (F_1(2) \oplus F_2(2) \oplus F_3(2)) \circ K_{2s+1} \\
  \quad = \{ F_1(2) \circ K_{2s+1} \} \oplus \{ F_2(2) \circ K_{2s+1} \} \oplus \{ F_3(2) \circ K_{2s+1} \} .
\]

Now \( 3(2s + 1)\hat{C}_{2m} \)-factors of \( G(2) \circ K_{2s+1} \) are constructed as follows, see Figure 2.3.

For \( 0 \leq k \leq 2s \),

\[
G_{k1} = \bigcup_{i=0}^{m-1} \{ \alpha_k(V_{2i}, V_{2i+1}) \} \oplus \bigcup_{i=0}^{m-3} \{ \alpha_k(V_{2i+1}, V_{2i+4}) \}
\quad \oplus \{ \alpha_k(V_0, V_2) \} \oplus \alpha_k(V_{2m-3}, V_{2m-1}),
\]

\[
G_{k2} = \bigcup_{i=0}^{m-2} \{ \alpha_k(V_{2i+1}, V_{2i+2}) \} \oplus \{ \alpha_k(V_0, V_{2m-1}) \} \oplus \bigcup_{i=0}^{m-3} \{ \alpha_k(V_{2i+1}, V_{2i+4}) \}
\quad \oplus \{ \alpha_k(V_0, V_2) \} \oplus \alpha_k(V_{2m-3}, V_{2m-1}) ,
\]

For \( 0 \leq k \leq s - 1 \),

\[
G_{k3} = \bigcup_{i=0}^{m-1} \{ \alpha_{k+1}(V_{2i}, V_{2i+1}) \} \oplus \bigcup_{i=0}^{m-1} \{ \alpha_{2s-k}(V_{2i+1}, V_{2i+2}) \} ,
\]

\[
G_{k4} = \bigcup_{i=0}^{m-1} \{ \alpha_{k+1}(V_{2i+1}, V_{2i+2}) \} \oplus \bigcup_{i=0}^{m-1} \{ \alpha_{2s-k}(V_{2i}, V_{2i+1}) \} ; \quad \text{and}
\]

\[
G_5 = \bigcup_{i=0}^{2m-1} \{ \alpha_0(V_i, V_{i+1}) \} .
\]

In all the above constructions the subscripts are taken modulo \( 2m - 1 \) with residues 0, 1, \ldots, \( 2m - 1 \). The structure of \( G \) in the hypothesis makes clear that each \( G_{ki} \), \( 1 \leq i \leq 4 \), and \( G_5 \) are \( \hat{C}_{2m} \)-factors of \( G(2) \circ K_{2s+1} \). Hence \( \hat{C}_{2m} \parallel G(2) \circ K_{2s+1} \). \( \square \)
Figure 2.3: $C_6$ $\parallel$ $G(2) \circ \overline{K}_3$, when $|G| = 6.$
(b) \( \{G_{12}, G_{22}, G_{03}, G_{04}\} \) of \( G(2) \circ \overline{K}_3 \).

Figure 2.3: \( \tilde{C}_6 \parallel G(2) \circ \overline{K}_3 \), when \( |G| = 6 \).
Figure 2.3: \( \hat{C}_6 \parallel G(2) \circ K_3 \), when \(|G| = 6\).

Theorem 2.3.13. \( \hat{C}_{2m} \parallel K_{2m}(2) \circ K_{2s+1} \). 

Proof. By Walecki’s construction, we have \( K_{2m} = H_1 \oplus H_2 \oplus \cdots \oplus H_{m-1} \oplus F \), where \( H_i \)’s are the Hamilton cycles and \( F \) is the 1-factor of \( K_{2m} \). Therefore

\[
K_{2m}(2) \circ K_{2s+1} = \{ H_1(2) \oplus H_2(2) \oplus \cdots \oplus H_{m-1}(2) \oplus F(2) \} \circ K_{2s+1} \\
= \{ H_1(2) \circ K_{2s+1} \} \oplus \cdots \oplus \{ H_{m-1}(2) \circ K_{2s+1} \} \\
\oplus \{ H_{m-1}(2) \oplus F(2) \} \circ K_{2s+1}.
\]

By Lemma 2.2.3 and Lemma 2.3.12, it is proved that \( \hat{C}_{2m} \parallel K_{2m}(2) \circ K_{2s+1} \).

Corollary 2.3.14. \( \hat{C}_{2m} \parallel K_{2m}(2\lambda) \circ K_{2s+1} \).

Theorem 2.3.15. \( \hat{C}_{2m} \parallel K_{2m}^*(2\lambda) \circ K_{2s+1} \).

Remark 2.3.16. Theorems 2.3.11 and 2.3.15 implies the existence of a resolvable \((k,n,k,2\lambda)\) multipartite \( \hat{C}_k \)-design for even \( k \).
2.4 Gregarious Cycle Factorizations of Tensor Product of Complete Graphs

In this section, the existence of $\hat{C}_m$-factorization of symmetric digraph of tensor product of complete graphs have been proved.

Lemma 2.4.1. $\hat{C}_{2m} \parallel (G \times K_2)^*$, where $G$ is the graph described in Remark 2.2.2.

Proof. Let $V(G) = \{v_0, v_1, \ldots, v_{2m-1}\}$ and $V(K_2) = \{u_0, u_1\}$. Then $V(G \times K_2)^* = \bigcup_{i=0}^{2m-1} V_i$, where $V_i = \{v_i^0, v_i^1\}$.

Factorize $(G \times K_2)^*$ into $\hat{C}_{2m}$-factors as follows. Let

$$G_1 = \{(v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_6^0 v_7^0 \ldots v_{2m-6}^0 v_{2m-5}^0 v_{2m-4}^0 v_{2m-3}^0 v_{2m-2}^0 v_{2m-1}^0)},$$

$$G_2 = \{(v_0^1 v_1^1 v_2^1 v_3^1 v_4^1 v_5^1 v_6^1 v_7^1 \ldots v_{2m-6}^1 v_{2m-5}^1 v_{2m-4}^1 v_{2m-3}^1 v_{2m-2}^1 v_{2m-1}^1)},$$

$$G_3 = \{(v_0^1 v_1^1 v_2^1 v_3^1 v_4^1 v_5^1 v_6^1 v_7^1 \ldots v_{2m-6}^1 v_{2m-5}^1 v_{2m-4}^1 v_{2m-3}^1 v_{2m-2}^1 v_{2m-1}^1 v_{2m}^0 v_{2m+1}^0 \ldots v_{3m}^0)},$$

$$G_4 = \{(v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_6^0 v_7^0 \ldots v_{2m-6}^0 v_{2m-5}^0 v_{2m-4}^0 v_{2m-3}^0 v_{2m-2}^0 v_{2m-1}^0 v_{2m}^1 v_{2m+1}^1 \ldots v_{3m}^1)},$$

Case 1: $2m \equiv 0 \pmod{4}$.

$$G_3 = \{(v_0^1 v_1^1 v_2^1 v_3^1 v_4^1 v_5^1 v_6^1 v_7^1 \ldots v_{2m-6}^1 v_{2m-5}^1 v_{2m-4}^1 v_{2m-3}^1 v_{2m-2}^1 v_{2m-1}^1 v_{2m}^0 v_{2m+1}^0 \ldots v_{3m}^0)},$$

$$G_4 = \{(v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_6^0 v_7^0 \ldots v_{2m-6}^0 v_{2m-5}^0 v_{2m-4}^0 v_{2m-3}^0 v_{2m-2}^0 v_{2m-1}^0 v_{2m}^1 v_{2m+1}^1 \ldots v_{3m}^1)},$$

Case 2: $2m \equiv 2 \pmod{4}$.

$$G_3 = \{(v_0^1 v_1^1 v_2^1 v_3^1 v_4^1 v_5^1 v_6^1 v_7^1 \ldots v_{2m-6}^1 v_{2m-5}^1 v_{2m-4}^1 v_{2m-3}^1 v_{2m-2}^1 v_{2m-1}^1 v_{2m}^0 v_{2m+1}^0 \ldots v_{3m}^0)},$$

$$G_4 = \{(v_0^0 v_1^0 v_2^0 v_3^0 v_4^0 v_5^0 v_6^0 v_7^0 \ldots v_{2m-6}^0 v_{2m-5}^0 v_{2m-4}^0 v_{2m-3}^0 v_{2m-2}^0 v_{2m-1}^0 v_{2m}^1 v_{2m+1}^1 \ldots v_{3m}^1)},$$

Clearly each $G_i$, $1 \leq i \leq 3$, is a $\hat{C}_{2m}$-factor of $(G \times K_2)^*$, see Figure 2.4. Hence $\hat{C}_{2m} \parallel (G \times K_2)^*$. □
Figure 2.4: $\hat{C}_8$-factorization of $(G \times K_2)^*$, when $|V(G)| = 8.$

**Theorem 2.4.2.** For $m > 1$, $\hat{C}_{2m} \parallel (K_{2m} \times K_2)^*$.

**Proof.** By Walecki’s construction, Theorem 2.2.1, we have $K_{2m} = H_1 \oplus H_2 \oplus \cdots \oplus H_{m-1} \oplus F$, where $H_i$’s are Hamilton cycles and $F$ is a 1-factor of $K_{2m}$. Therefore

$$(K_{2m} \times K_2)^* = ((H_1 \oplus H_2 \oplus \cdots \oplus H_{m-1} \oplus F) \times K_2)^*$$

$$= (H_1 \times K_2)^* \oplus \cdots \oplus ((H_{m-1} \oplus F) \times K_2)^*.$$ 

By Theorem 2.2.5, $C_{2m} \parallel (H_i \times K_2)$. In this $C_{2m}$-factorization all the cycles of the factors meet all the $2m$ parts of the graph $H_i \times K_2$, see [59], and hence $\hat{C}_{2m} \parallel (H_i \times K_2)^*$, for $i = 1, 2, \ldots, m - 2$ and by Lemma 2.4.1, 

$\hat{C}_{2m} \parallel ((H_{m-1} \oplus F) \times K_2)^*$. Thus $\hat{C}_{2m} \parallel (K_{2m} \times K_2)^*$.

**Theorem 2.4.3.** For $m \geq 2$, $\hat{C}_{2m} \parallel (K_{2m} \times K_{2r})^*$. 


Proof. Let \( \{F_1, F_2, \ldots, F_{2r-1}\} \) be a 1-factorization of \( K_{2r} \) and hence

\[
(K_{2m} \times K_{2r})^* = (K_{2m} \times (F_1 \oplus F_2 \oplus \cdots \oplus F_{2r-1}))^*.
\]

Then, \( (K_{2m} \times K_{2r})^* = \bigoplus_{i=1}^{2r-1} (K_{2m} \times F_i)^* \).

As each \( (K_{2m} \times F_i)^* \approx r (K_{2m} \times K_2)^* \),
by Theorem 2.4.2, we have \( \tilde{\mathcal{C}}_{2m} || (K_{2m} \times F_i)^* \).

Hence \( \tilde{\mathcal{C}}_{2m} || (K_{2m} \times K_{2r})^* \). \[\square\]

As the tensor product is distributive over edge-disjoint subgraphs, we have the following.

**Corollary 2.4.4.** For \( m \geq 2 \), \( \tilde{\mathcal{C}}_{2m} \parallel (K_{2m}(\lambda) \times K_{2r}(\mu))^* \). \[\square\]

**Lemma 2.4.5.** For \( m \geq 2 \), \( \tilde{\mathcal{C}}_{2m} \parallel (G \times K_{2r+1})^* \), where \( G \) is the graph described in Remark 2.2.2.

**Proof.** Let \( V(G) = \{v_0, v_1, \ldots, v_{2m-1}\} \) and let \( V(K_{2r+1}) = \{u_0, u_1, \ldots, u_r\} \).

Then \( V(G \times K_{2r+1})^* = \bigcup_{i=0}^{2m-1} V_i \), where \( V_i = \{v_i^0, v_i^1, \ldots, v_i^{2r}\} \).

By Theorem 2.2.1, \( (G \times K_{2r+1})^* = (G \times (H_1 \oplus \cdots \oplus H_r))^* \),
where \( \{H_1, \ldots, H_r\} \) is a Hamilton cycle decomposition of \( K_{2r+1} \). Then

\( (G \times K_{2r+1})^* = \bigoplus_{i=1}^r (G \times H_i)^* \). As each \( (G \times H_i)^* \approx (G \times C_{2r+1})^* \), it is enough to show \( \tilde{\mathcal{C}}_{2m} \parallel (G \times C_{2r+1})^* \). \( \tilde{\mathcal{C}}_{2m} \)-factorization of \( (G \times C_{2r+1})^* \) is described as follows. Let

\[
\begin{align*}
G_1 &= \bigcup_{i=0}^{m-1} \alpha_1(V_{2i}, V_{2i+1}) \oplus \bigcup_{i=0}^{m-1} \alpha_2(V_{2i+1}, V_{2i+2}), \\
G_2 &= \bigcup_{i=0}^{m-1} \alpha_2(V_{2i+2}, V_{2i+1}) \oplus \bigcup_{i=0}^{m-1} \alpha_1(V_{2i+1}, V_{2i}) \\
G_3 &= \bigcup_{i=0}^{m-3} \alpha_2(V_{2i+4}, V_{2i+1}) \oplus \alpha_2(V_2, V_0) \oplus \alpha_2(V_{2m-1}, V_{2m-3})
\end{align*}
\]
The remaining edges of \((G \times C_{2r+1})^*\) are partitioned into \(G_5\) and \(G_6\) according as \(m\) is odd or even as follows.

**Case 1: \(2m \equiv 0 \pmod{4}\)**

\[G_5 = \alpha_1 (V_0, V_2) \oplus \bigcup_{i=1,3}^{m-3} (\alpha_2r (V_{2i-1}, V_{2i+1}) \oplus \alpha_1 (V_0, V_{2m-3}))\]

\[\oplus \alpha_2r (V_{2m-2}, V_{2m-4}) \oplus \alpha_1 (V_{2m-2}, V_{2m-4})\]

\[\oplus \bigcup_{i=1,3}^{m-3} (\alpha_2r (V_{2m-2i-1}, V_{2m-2i-2}) \oplus \alpha_1 (V_{2m-2i-2}, V_{2m-2i-4}))\]

\[\oplus \alpha_2r (V_1, V_0)\]

\[G_6 = \alpha_2r (V_0, V_1) \oplus \bigcup_{i=1,3}^{m-3} (\alpha_1 (V_{2i-1}, V_{2i+2}) \oplus \alpha_2r (V_{2i+2}, V_{2i+3}))\]

\[\oplus \alpha_1 (V_{2m-3}, V_{2m-1}) \oplus \alpha_2r (V_{2m-1}, V_{2m-3})\]

\[\oplus \bigcup_{i=1,3}^{m-3} (\alpha_1 (V_{2m-2i-1}, V_{2m-2i-3}) \oplus \alpha_2r (V_{2m-2i-3}, V_{2m-2i-4}))\]

\[\oplus \alpha_1 (V_2, V_0)\]

**Case 2: \(2m \equiv 2 \pmod{4}\)**

\[G_5 = \alpha_1 (V_0, V_2) \oplus \bigcup_{i=1,3}^{m-4} (\alpha_2r (V_{2i}, V_{2i+1}) \oplus \alpha_1 (V_0, V_{2m-1}))\]

\[\oplus \alpha_2r (V_{2m-4}, V_{2m-3}) \oplus \alpha_1 (V_{2m-3}, V_{2m-1})\]

\[\oplus \bigcup_{i=1,3}^{m-2} (\alpha_2r (V_{2m-2i+1}, V_{2m-2i}))\]

\[\oplus \alpha_1 (V_{2m-2}, V_{2m-2i-1}) \oplus \alpha_2r (V_1, V_0)\]

\[G_6 = \alpha_2r (V_0, V_1) \oplus \bigcup_{i=1,3}^{m-2} (\alpha_1 (V_{2i-1}, V_{2i+2}) \oplus \alpha_2r (V_{2i+2}, V_{2i+3}))\]
\[ \oplus \alpha_1 (V_{2m-1}, V_{2m-3}) \oplus \bigcup_{i=1,3}^{m-4} (\alpha_{2r} (V_{2m-2i-1}, V_{2m-2i-2}) \oplus \alpha_1 (V_{2m-2i-2}, V_{2m-2i-5})) \oplus \alpha_{2r} (V_3, V_2) \oplus \alpha_1 (V_2, V_0). \]

Figure 2.5: \{G_1, G_2, G_3, G_4\} of \((G \times K_5)^*\)
The structure of $G$ makes clear that, each $G_i$, $1 \leq i \leq 6$, is a $\hat{C}_{2m}$-factor of $(G \times C_{2r+1})^*$, see Figure 2.5. Thus $\hat{C}_{2m} \parallel (G \times C_{2r+1})^*$, and hence $\hat{C}_{2m} \parallel (G \times K_{2r+1})^*$. \hfill $\square$

**Theorem 2.4.6.** For $m \geq 2$, $\hat{C}_{2m} \parallel (K_{2m} \times K_{2r+1})^*$.

**Proof.** By Walecki’s construction, Theorem 2.2.1, we have

$$K_{2m} = H_1 \oplus H_2 \oplus \cdots \oplus H_{m-1} \oplus F,$$

where $\{H_1, H_2, \ldots, H_{m-1}, F\}$ is a Hamilton cycle decomposition of $K_{2m}$. Therefore

$$K_{2m} \times K_{2r+1}^* = ((H_1 \oplus \cdots \oplus H_{m-1} \oplus F) \times K_{2r+1})^*$$

$$= (H_1 \times K_{2r+1})^* \oplus \cdots \oplus ((H_{m-1} \oplus F) \times K_{2r+1})^*.$$
By Theorem 2.2.5,

\[ C_{2m} \parallel H_i \times K_{2r+1}, \quad \text{and hence} \]

\[ \hat{C}_{2m} \parallel (H_i \times K_{2r+1})^* \quad \text{for } i = 1, 2, \ldots, m - 2. \quad \text{By Lemma 2.4.5,} \]

\[ \hat{C}_{2m} \parallel ((H_{m-1} \oplus F) \times K_{2r+1})^*. \]

Hence \( \hat{C}_{2m} \parallel (K_{2m} \times K_{2r+1})^*. \) \( \square \)

Because of distributivity of the tensor product over edge-disjoint subgraphs, we have the following.

**Corollary 2.4.7.** For \( m \geq 2, \) \( \hat{C}_{2m} \parallel (K_{2m} (\lambda) \times K_{2r+1} (\mu))^*. \) \( \square \)

**Lemma 2.4.8.** For \( n \neq 2, 6, \) \( \hat{C}_3 \parallel C_3 \circ K_n. \)

**Proof.** Let \( L_1 \) and \( L_2 \) be mutually orthogonal latin squares [MOLS] of order \( n \) (such a pair exists by Lemma 2.2.6). By superimposing \( L_2 \) on \( L_1 \), there is an array \( L_3 \) in which every entry is an ordered pair of elements from \( L_1 \) and \( L_2 \). Since \( L_1 \) and \( L_2 \) are MOLS, \( L_3 \) contains all the \( n^2 \) ordered pairs. For a fixed \( k \), mark all the \( n \) entries \( (k, l), l = 0, 1, \ldots, n - 1 \) in \( L_3 \). If \( (k, l) \) is the \((i, j)^{th}\) entry of \( L_3 \) then \( v_i^k v_j^l v_j^l v_i^k \) is a 3-cycle in \( C_3 \circ K_n \). Therefore when \( l \) varies from 0, 1, \ldots, \( n - 1 \) we get a \( \hat{C}_3 \)-factor of \( C_3 \circ K_n \). Also when \( k \) varies from 0, 1, \ldots, \( n - 1 \) we get a \( \hat{C}_3 \)-factorization of \( C_3 \circ K_n \), for example see Figure 2.6. \( \square \)
Figure 2.6: $\hat{C}_3$-decomposition of $C_3 \circ K_3$

**Example:** For $n = 3$

$$
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 1
\end{array}
\begin{array}{ccc}
1 & 2 & 0 \\
0 & 1 & 2 \\
2 & 0 & 1
\end{array}
\begin{array}{ccc}
(0,1) & (1,2) & (2,0) \\
(1,0) & (2,1) & (0,2) \\
(2,2) & (0,0) & (1,1)
\end{array}

Theorem 2.4.9. For $n \neq 2, 6$, $\hat{C}_{2m+1} \parallel K_{2m+1} \circ \overline{K}_n$.

**Proof.** Let $\{V_0, V_1, \ldots, V_{2m}\}$ be the vertex sub sets of the graph $K_{2m+1} \circ \overline{K}_n$.

Obtain $L_1$, $L_2$ and $L_3$ as in Lemma 2.4.8. If $(k, l)$ is the $(i, j)$th entry of $L_3$ then we have $v_0^k v_1^l v_2^d v_0^d$, a $\hat{C}_3$ in $C_3 \circ \overline{K}_n$. When $k$ and $l$ varies from 1, 2, 3, ..., $n$ we get a $\hat{C}_3$-factorization of $C_3 \circ \overline{K}_n$ as in the proof of Lemma 2.4.8. Now by extending this 3-cycle $v_0^k v_1^l v_2^d v_0^d$, we have $v_0^k v_1^l v_2^d v_3^d v_4^d v_5^d v_6^d v_7^d \ldots v_{2m-1}^d v_{2m}^d v_0^d$, a $\hat{C}_{2m+1}$ in $C_{2m+1} \circ \overline{K}_n$. In this fashion, by increasing the length of all cycles in the $\hat{C}_3$-factors of $C_3 \circ \overline{K}_n$ to $2m+1$, we get all the $\hat{C}_{2m+1}$-factors of $C_{2m+1} \circ \overline{K}_n$. 
That is \( \hat{C}_{2m+1} \parallel C_{2m+1} \circ \overline{K}_n \). We know that
\[
K_{2m+1} \circ \overline{K}_n = (H_1 \oplus \ldots \oplus H_m) \circ \overline{K}_n = \oplus_{i=1}^{m} (H_i \circ \overline{K}_n),
\]
where \( \{H_1, H_2, \ldots, H_m\} \) is a Hamilton cycle decomposition of \( K_{2m+1} \). As each \( H_i \circ \overline{K}_n \cong C_{2m+1} \circ \overline{K}_n \), we have \( \hat{C}_{2m+1} \parallel K_{2m+1} \circ \overline{K}_n \). \( \square \)

**Corollary 2.4.10.** For \( n \neq 2,6 \), \( \hat{C}_{2m+1} \parallel (K_{2m+1} (\lambda) \circ \overline{K}_n)^* \). \( \square \)

**Lemma 2.4.11.** For \( n \neq 2,6 \), \( \hat{C}_{2m+1} \parallel C_{2m+1} \times K_n \).

**Proof.** Rearrange the vertices of \( C_{2m+1} \circ \overline{K}_n \) so that the edges of one \( \hat{C}_{2m+1} \)-factor of \( C_{2m+1} \circ \overline{K}_n \) become, say, \( \cup_{l=0}^{l} \alpha_0 (V_i, V_i) \). By removing such 2-factor from \( C_{2m+1} \circ \overline{K}_n \) we get the graph \( C_{2m+1} \times K_n \), since \( C_{2m+1} \times K_n = (C_{2m+1} \circ \overline{K}_n) \setminus \{ \cup_{l=0}^{l} \alpha_0 (V_i, V_i) \} \). The remaining \( \hat{C}_{2m+1} \)-factors of \( C_{2m+1} \circ \overline{K}_n \) factorizes \( C_{2m+1} \times K_n \) and hence \( \hat{C}_{2m+1} \parallel C_{2m+1} \times K_n \). \( \square \)

As the tensor product is distributive over edge-disjoint subgraphs and \( K_{2m+1} \) has a Hamilton cycle decomposition, we have the following.

**Theorem 2.4.12.** For \( n \neq 2,6 \), \( \hat{C}_{2m+1} \parallel K_{2m+1} \times K_n \). \( \square \)

**Corollary 2.4.13.** For \( n \neq 2,6 \), \( \hat{C}_{2m+1} \parallel (K_{2m+1} (\lambda) \times K_n (\mu))^* \). \( \square \)

**Note 2.1.** By Theorem 2.2.7 and its proof adopted therein, for \( m > 1 \), \( C_{2m+1} \parallel C_{2m+1} \circ \overline{K}_6 \) and hence by adopting the technique employed in the proof of Lemma 2.4.11, we have \( \hat{C}_{2m+1} \parallel (C_{2m+1} \times K_6) \).

Thus \( \hat{C}_{2m+1} \parallel K_{2m+1} \times K_6 \).
since $K_{2m+1} \times K_6 = (H_1 \oplus \cdots \oplus H_m) \times K_6 = \bigoplus_{i=1}^{m} (H_i \times K_6),
where \{H_1, H_2, \ldots, H_m\}$ is a Hamilton cycle decomposition of $K_{2m+1}$.

\[\text{Theorem 2.4.14. For all } n > 2, m \geq 3 \text{ and } (m, n) \neq (3, 6),
\hat{C}_m \parallel (K_m \times K_6)^*.\]

\[\text{Proof. By Note 2.1, Theorems 2.4.3, 2.4.6, and 2.4.12 the result follows. \quad \square}\]

\[\text{Lemma 2.4.15. } \hat{C}_3 \parallel (K_3 \times K_6)^*.\]

\[\text{Proof. Factorize } (K_3 \times K_6)^* \text{ into } \hat{C}_3 \text{ as follows: Let}\]

\[H_1 = (v_0^0 v_1^0 v_2^1) \oplus (v_0^0 v_1^1 v_2^1) \oplus (v_0^0 v_1^2 v_2^0) \oplus (v_0^0 v_1^2 v_2^1) \oplus (v_0^0 v_1^1 v_2^0),\]
\[H_2 = (v_0^0 v_1^2 v_2^1) \oplus (v_0^0 v_1^0 v_2^0) \oplus (v_0^0 v_1^1 v_2^0) \oplus (v_0^0 v_1^0 v_2^1) \oplus (v_0^0 v_1^0 v_2^2),\]
\[H_3 = (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2),\]
\[H_4 = (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2),\]
\[H_5 = (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2),\]
\[H_6 = (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2),\]
\[H_7 = (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2),\]
\[H_8 = (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2),\]
\[H_9 = (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2),\]
\[H_{10} = (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2) \oplus (v_0^0 v_1^0 v_2^2).\]

Clearly each $H_i, 1 \leq i \leq 10$, is a $\hat{C}_3$-factor of $(K_3 \times K_6)^*$.

Hence $\hat{C}_3 \parallel (K_3 \times K_6)^*$, see Figure 2.7. \quad \square
Figure 2.7: $\tilde{C}_3 \parallel (K_3 \times K_6)^*$. 
**Theorem 2.4.16.** For all $n > 2$, $m \geq 3$, $\hat{C}_m \parallel (K_m \times K_n)^*$.

**Proof.** Proof follows from Theorem 2.4.14 and Lemma 2.4.15. □

As the tensor product is distributive over edge-disjoint subgraphs, we have the following.

**Corollary 2.4.17.** For all $n > 2$, $m \geq 3$, $\hat{C}_m \parallel (K_m(\lambda) \times K_n(\mu))^*$. □