CHAPTER III

\textit{\(\omega\)-CONTINUOUS MAPS IN TOPOLOGICAL SPACES}

3.1. Introduction

Several authors (\cite{1}, \cite{4}, \cite{20}, \cite{22}, \cite{23}, \cite{33}, \cite{34}, \cite{36}, \cite{37}, \cite{49}, \cite{61}, \cite{72}, \cite{73}, \cite{79} and \cite{95}) working in the field of general topology have shown more interest in studying the concepts of generalizations of continuous maps. A weak form of continuous maps called \(g\)-continuous maps was introduced by Balachandran, Sundaram and Maki \cite{10}. Recently Dontchev \cite{30} and Veera Kumar \cite{121} introduced and studied another form of generalized continuous maps called \(gsp\)-continuous maps and \(g^*\)-continuous maps respectively.

In this chapter we first introduce \(\omega\)-continuous maps and study their relations with various generalized continuous maps. We also discuss some properties of \(\omega\)-continuous maps and obtain the pasting lemma for \(\omega\)-continuous maps. We then introduce \(\omega\)-irresolute maps, strongly \(\omega\)-continuous maps and perfectly \(\omega\)-continuous maps in topological spaces and discuss some of their properties.

Finally using \(\omega\)-continuous maps we obtain a decomposition of continuity. As another application of these maps, we construct two new sandwich type near rings.
3.2. $\omega$-continuous maps

In this section we introduce $\omega$-continuous maps in topological spaces and we prove that the composition of two $\omega$-continuous maps need not be $\omega$-continuous. Also, we generalize the pasting lemma for $\omega$-continuous maps. At the end of this section we explore certain characterizations of $\omega$-continuous maps.

**Definition 3.2.1** A map $f: (X, \tau) \to (Y, \sigma)$ is called $\omega$-continuous if $f^{-1}(F)$ is $\omega$-closed in $(X, \tau)$ for every closed set $F$ in $(Y, \sigma)$.

**Example 3.2.2** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$, $Y = \{p, q\}$ and $\sigma$ be the discrete topology on $Y$. Define $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = f(b) = p$ and $f(c) = q$, then $f$ is $\omega$-continuous, because every subset of $(X, \tau)$ is $\omega$-closed.

**Proposition 3.2.3** Every continuous map is $\omega$-continuous but not conversely.

The proof follows from Proposition 2.2.2.

**Example 3.2.4** The function $f$ in Example 3.2.2 is $\omega$-continuous but not continuous, because for the open set $U = \{q\}$ in $(Y, \sigma)$, $f^{-1}(U) = \{c\}$, which is not open in $(X, \tau)$.

Thus the class of all $\omega$-continuous maps properly contains the class of all continuous maps. In the next propositions we show that the class of all $\omega$-continuous maps is properly contained in the classes of various generalized continuous maps in topological spaces.

**Proposition 3.2.5** Every $\omega$-continuous map is g-continuous but not conversely.

The proof follows from Proposition 2.2.4.
Example 3.2.6 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $(Y, \sigma)$ be the topological space of Example 3.2.2. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = f(b) = p$ and $f(c) = q$. Then $f$ is g-continuous but not $\omega$-continuous.

Proposition 3.2.7 Every $\omega$-continuous map is sg-continuous and hence $\beta$-continuous but not conversely. Also every $\omega$-continuous maps is $g\alpha$-continuous and hence pre-continuous but not conversely.

The proof follows from Propositions 2.2.6 and 2.2.7.

Example 3.2.8 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then $f$ is sg-continuous, $\beta$-continuous, $g\alpha$-continuous and pre-continuous but not $\omega$-continuous.

Proposition 3.2.9 Every $\omega$-continuous map is gs-continuous, gsp-continuous, rg-continuous, gp-continuous, gpr-continuous, wg-continuous, $\alpha g$-continuous and hence $\alpha^{**} g$-continuous but not conversely.

The proof follows from Proposition 2.2.8.

The map $f$ in Example 3.2.8 is gs-continuous, gsp-continuous, rg-continuous, gp-continuous, gpr-continuous, wg-continuous, $\alpha g$-continuous and hence $\alpha^{**} g$-continuous. But $f$ is not $\omega$-continuous.

Remark 3.2.10 When $(X, \tau)$ is T$_\omega$ (resp. gT$_\omega$, $\alpha T_\omega$, w$_g$T$_\omega$), the concepts of continuity (resp. g-continuity, $\alpha g$-continuity and wg-continuity) and $\omega$-continuity coincide.

Remark 3.2.11 The following Examples show that $\omega$-continuity is independent of g*-continuity, $\alpha$-continuity, semi-continuity, $\theta$-g-continuity and $\delta$-g-continuity.

Example 3.2.12 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then $f$ is g*-continuous, $\theta$-g-continuous and $\delta$-g-continuous but not $\omega$-continuous.
The map \( f \) in Example 3.2.8 is both \( \alpha \)-continuous and semi-continuous but not \( \omega \)-continuous.

**Example 3.2.13** Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b, c\}, X\} \) and \( (Y, \sigma) \) be the topological space of Example 3.2.2. Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = f(c) = p \) and \( f(b) = q \). Then \( f \) is \( \omega \)-continuous but neither \( g^* \)-continuous nor \( \alpha \)-continuous nor semi-continuous.

**Example 3.2.14** Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \) and \( Y = \{p, q\} \) and \( \sigma = \{\emptyset, \{p\}, Y\} \). Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = f(b) = p \) and \( f(c) = q \). Then \( f \) is neither \( \theta \)-\( g \)-continuous nor \( \delta \)-\( g \)-continuous. However \( f \) is \( \omega \)-continuous.

**Proposition 3.2.15** A map \( f: (X, \tau) \to (Y, \sigma) \) is \( \omega \)-continuous if and only if \( f^{-1}(U) \) is \( \omega \)-open in \( (X, \tau) \) for every open set \( U \) in \( (Y, \sigma) \).

**Proof:** Let \( f: (X, \tau) \to (Y, \sigma) \) be \( \omega \)-continuous and \( U \) be an open set in \( (Y, \sigma) \). Then \( U^c \) is closed in \( (Y, \sigma) \) and since \( f \) is \( \omega \)-continuous, \( f^{-1}(U^c) \) is \( \omega \)-closed in \( (X, \tau) \). But \( f^{-1}(U^c) = (f^{-1}(U))^c \) and so \( f^{-1}(U) \) is \( \omega \)-open in \( (X, \tau) \).

Conversely, assume that \( f^{-1}(U) \) is \( \omega \)-open in \( (X, \tau) \) for each open set \( U \) in \( (Y, \sigma) \). Let \( F \) be a closed set in \( (Y, \sigma) \). Then \( F^c \) is open in \( (Y, \sigma) \) and by assumption, \( f^{-1}(F^c) \) is \( \omega \)-open in \( (X, \tau) \). Since \( f^{-1}(F^c) = (f^{-1}(F))^c \), we have \( f^{-1}(F) \) is closed in \( (X, \tau) \) and so \( f \) is \( \omega \)-continuous.

**Proposition 3.2.16** A map \( f: (X, \tau) \to (Y, \sigma) \) is \( \omega \)-continuous if and only if \( f: (X, \tau^o) \to (Y, \sigma) \) is continuous.

**Proof:** Assume that \( f: (X, \tau) \to (Y, \sigma) \) is \( \omega \)-continuous. Then \( f^{-1}(U) \in \tau^o \) for every \( U \in \sigma \). Therefore, \( f: (X, \tau^o) \to (Y, \sigma) \) is continuous.

Conversely, assume that \( f: (X, \tau^o) \to (Y, \sigma) \) is continuous. Then \( f^{-1}(G) \in \tau^o \) for every \( G \in \sigma \). Therefore, \( f: (X, \tau) \to (Y, \sigma) \) is \( \omega \)-continuous.

**Remark 3.2.17** The composition of two \( \omega \)-continuous maps need not be \( \omega \)-continuous and this is shown by the following example.
Example 3.2.18 Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$ and $\eta = \{\emptyset, \{a, c\}, Z\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = f(c) = c$, $f(b) = b$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be the identity map. Then $f$ and $g$ are $\omega$-continuous but their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is not $\omega$-continuous, because $F = \{b\}$ is closed in $(Z, \eta)$ but $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F)) = f^{-1}(\{b\}) = \{b\}$, which is not $\omega$-closed in $(X, \tau)$.

Proposition 3.2.19 Let $(X, \tau)$ and $(Z, \eta)$ be topological spaces and $(Y, \sigma)$ be a $T_\omega$-space. Then the composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ of the $\omega$-continuous maps $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is $\omega$-continuous.

Proof: Let $F$ be any closed set of $(Z, \sigma)$. Then $g^{-1}(F)$ is closed in $(Y, \sigma)$, since $g$ is $\omega$-continuous and $(Y, \sigma)$ is a $T_\omega$-space. Since $g^{-1}(F)$ is closed in $(Y, \sigma)$ and $f$ is $\omega$-continuous, $f^{-1}(g^{-1}(F))$ is $\omega$-closed in $(X, \tau)$. But $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ and so $g \circ f$ is $\omega$-continuous.

Proposition 3.2.20 Let $(X, \tau)$ and $(Z, \eta)$ be any topological spaces and $(Y, \sigma)$ be a $T_{1/2}$ space (resp. $T_b$ space, a$T_b$ space). Then the composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ of the $\omega$-continuous map $f : (X, \tau) \rightarrow (Y, \sigma)$ and the $g$-continuous (resp. $g_s$-continuous, $\alpha g$-continuous) map $g : (Y, \sigma) \rightarrow (Z, \eta)$ is $\omega$-continuous.

Proof: Similar to Proposition 3.2.19.

Proposition 3.2.21 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\omega$-continuous and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is continuous, then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is $\omega$-continuous.

Proof: Let $F$ be any closed set in $(Z, \eta)$. Since $g : (Y, \sigma) \rightarrow (Z, \eta)$ is continuous, $g^{-1}(F)$ is closed in $(Y, \sigma)$. Since $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\omega$-continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\omega$-closed in $(X, \tau)$ and so $g \circ f$ is $\omega$-continuous.
We have the following proposition for the restriction of a \( \omega \)-continuous map.

**Proposition 3.2.22** Let \( f : (X, \tau) \to (Y, \sigma) \) be an \( \omega \)-continuous map and let \( H \) be an \( \omega \)-closed subset of \( (X, \tau) \). Then the restriction \( f_H : (H, \tau_H) \to (Y, \sigma) \) is also \( \omega \)-continuous.

**Proof:** Let \( F \) be any closed set in \((Y, \sigma)\). Since \( f \) is \( \omega \)-continuous, \( f^{-1}(F) \) is \( \omega \)-closed in \((X, \tau)\). Let \( f^{-1}(F) \cap H = H_1 \). Then \( H_1 \) is \( \omega \)-closed in \((X, \tau)\) by Corollary 2.2.21. Since \( (f_H)^{-1}(F) = f^{-1}(F) \cap H = H_1 \), we need to show that \( H_1 \) is \( \omega \)-closed in \((H, \tau_H)\). Let \( U \) be any semi-open set of \((H, \tau_H)\) such that \( H_1 \subseteq U \). Since \( U \) is semi-open in \((H, \tau_H)\), \( U = G \cap H \) for some semi-open set \( G \) in \((X, \tau)\) by Lemma 2.2.38. Now, \( H_1 \subseteq G \cap H \) and so \( H_1 \subseteq G \). Since \( H_1 \) is \( \omega \)-closed in \((X, \tau)\), \( \text{cl}(H_1) \subseteq G \). We have \( \text{cl}_H(H_1) = \text{cl}(H_1) \cap H \subseteq G \cap H = U \) and therefore \( H_1 \) is \( \omega \)-closed in \((H, \tau_H)\) and hence \( f_H \) is \( \omega \)-continuous.

**Theorem 3.2.23** Let \( f : (X, \tau) \to (Y, \sigma) \) be a mapping and \( \{A_\alpha : \alpha \in \Lambda\} \) a \( \omega \)-open cover of \( X \), that is \( A_\alpha \in \tau^\omega \) for each \( \alpha \in \Lambda \) and \( X = \cup_{\alpha \in \Lambda} A_\alpha \). If the restriction \( f_{A_\alpha} : (A_\alpha, \tau_{A_\alpha}) \to (Y, \sigma) \) is \( \omega \)-continuous for each \( \alpha \in \Lambda \), then \( f \) is \( \omega \)-continuous.

**Proof:** Let \( U \) be an arbitrary open set in \((Y, \sigma)\). Now, for each \( \alpha \in \Lambda \), we have \( (f_{A_\alpha})^{-1}(U) = f^{-1}(U) \cap A_\alpha \). Then \( f^{-1}(U) \cap A_\alpha \) is \( \omega \)-open in \( A_\alpha \), since \( f_{A_\alpha} \) is \( \omega \)-continuous for each \( \alpha \in \Lambda \). By Proposition 2.4.8, \( f^{-1}(U) \cap A_\alpha \in \tau^\omega \) for each \( \alpha \in \Lambda \) and by Proposition 2.4.4, \( \cup_{\alpha \in \Lambda} f^{-1}(U) \cap A_\alpha = f^{-1}(U) \in \tau^\omega \). Therefore \( f \) is \( \omega \)-continuous.

Next we obtain the pasting lemma for \( \omega \)-continuous maps analogous to \( \alpha \)-continuous maps [66].

**Definition 3.2.24** Let \( \{A_i : i \in \Lambda\} \) be a given cover of a set \( A \) and \( \{f_i : i \in \Lambda\} \) be a family of maps \( f_i : A_i \to Z \) from \( A_i \) to a set \( Z \). We say that the
maps \( f_i \) are compatible if for every pair \( i, j \in \Lambda \), we have \( f_i(A_i \cap A_j) = f_j(A_i \cap A_j) \).

Then we define a map \( \nabla f : \Lambda \to Z \) as follows:

\[
(\nabla f)(x) = f_i(x) \quad \text{for every } x \in A_i.
\]

This is called the combination of the maps \( \{f_i\}_{i \in \Lambda} \) and this is obtained by pasting \( \{f_i\}_{i \in \Lambda} \) together with their common domains.

If \( \Lambda = \{1, 2, 3, \ldots, k\} \), then this is denoted by \( f_1 \nabla f_2 \nabla \cdots \cdots \nabla f_k \).

**Theorem 3.2.25 (Pasting lemma for \( \omega \)-continuous maps)**

Let \((X, \tau)\) and \((Y, \sigma)\) be any topological spaces.

i) Let \( \{A, B\} \) be an \( \omega \)-closed and open cover of \((X, \tau)\) and let \( \{f, g\} \) be a family of compatible \( \omega \)-continuous maps, where \( f : (A, \tau_A) \to (Y, \sigma) \) and \( g : (B, \tau_B) \to (Y, \sigma) \). Then, the combination \( f \nabla g : (X, \tau) \to (Y, \sigma) \) is \( \omega \)-continuous.

ii) Let \( \{A_i : i \in \Lambda\} \) be an \( \omega \)-open cover of \((X, \tau)\) and let \( \{f_i : i \in \Lambda\} \) be a family of compatible \( \omega \)-continuous maps \( f_i : (A_i, \tau_{A_i}) \to (Y, \sigma) \). Then the combination \( \nabla f : (X, \tau) \to (Y, \sigma) \) is \( \omega \)-continuous.

**Proof:**

i). Let \( F \) be any closed set in \((Y, \sigma)\). Then \( (f \nabla g)^{-1}(F) = [(f \nabla g)^{-1}(F) \cap A] \cup [(f \nabla g)^{-1}(F) \cap B] = f^{-1}(F) \cup g^{-1}(F) = C \cup D \) where \( C = f^{-1}(F) \) and \( D = g^{-1}(F) \). Since \( f \) is \( \omega \)-continuous, \( C \) is \( \omega \)-closed in \((A, \tau_A)\). By hypothesis \( A \) is \( \omega \)-closed and open in \((X, \tau)\). Therefore by Preposition 2.2.35, \( C \) is \( \omega \)-closed in \((X, \tau)\). Similarly \( D \) is \( \omega \)-closed in \((X, \tau)\) and by Proposition 2.2.14, \( (f \nabla g)^{-1}(F) \) is \( \omega \)-closed in \((X, \tau)\). Hence \( f \nabla g \) is \( \omega \)-continuous.

ii). Let \( U \) be an open set of \((X, \tau)\) and let \( f = \nabla f \). Then \( f^{-1}(U) = \bigcup_{i \in \Lambda} f_i^{-1}(U) \cap A_i \) = \( \bigcup_{i \in \Lambda} f_i^{-1}(U) = \bigcup_{i \in \Lambda} G_i \), where \( G_i = f_i^{-1}(U) \). For each \( i \in \Lambda \), \( G_i \) is \( \omega \)-open in \((A_i, \tau_{A_i})\). Since \( A_i \) is \( \omega \)-open in \((X, \tau)\) for each \( i \), by Proposition 2.4.8, \( G_i \) is \( \omega \)-open in \((X, \tau)\) for each \( i \). Hence \( f^{-1}(U) \) is \( \omega \)-open in \((X, \tau)\) by Proposition 2.4.4 and thus \( \nabla f \) is \( \omega \)-continuous.
Proposition 3.2.26  If A is $\omega$-closed in $(X, \tau)$ and if $f : (X, \tau) \to (Y, \sigma)$ is irresolute and closed, then $f(A)$ is $\omega$-closed in $(Y, \sigma)$.

Proof: Let $U$ be any semi-open in $(Y, \sigma)$ such that $f(A) \subseteq U$. Then $A \subseteq f^{-1}(U)$ and by hypothesis, $\text{cl}(A) \subseteq f^{-1}(U)$. Thus $f(\text{cl}(A)) \subseteq U$ and $f(\text{cl}(A))$ is a closed set. Now, $\text{cl}(f(A)) \subseteq \text{cl}(f(\text{cl}(A))) = f(\text{cl}(A)) \subseteq U$. i.e., $\text{cl}(f(A)) \subseteq U$ and so $f(A)$ is $\omega$-closed.

Remark 3.2.27 Under closed irresolute maps, $\omega$-open sets are generally not taken into $\omega$-open sets as seen from the following example.

Example 3.2.28  Let $X = \{a\}$, $\tau = \{\emptyset, X\}$, $Y = \{p, q\}$ and $\sigma = \{\emptyset, \{p\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$. This map is both irresolute and closed map. Let $A = X = \{a\}$. Then $A$ is $\omega$-open but $f(A) = \{q\}$, which is not $\omega$-open in $(Y, \sigma)$.

Theorem 3.2.29  Let $f : (X, \tau) \to (Y, \sigma)$ be a pre-semi-closed and open bijection and if $(X, \tau)$ is a $T_\omega$ space, then $(Y, \sigma)$ is also a $T_\omega$ space.

Proof: Let $y \in Y$. Since $f$ is bijective, $y = f(x)$ for some $x \in X$. By hypothesis $(X, \tau)$ is a $T_\omega$ space and so $\{x\}$ is semi-closed or open by Theorem 2.5.11. If $\{x\}$ is semi-closed then $\{y\} = f(\{x\})$ is semi-closed, since $f$ is pre-semi-closed. Also $\{y\}$ is open if $\{x\}$ is open since $f$ is open. Therefore by Theorem 2.5.11, $(Y, \sigma)$ is a $T_\omega$ space.

Theorem 3.2.30  If $f : (X, \tau) \to (Y, \sigma)$ is $\omega$-continuous and pre-semi-closed and if $A$ is an $\omega$-open (or $\omega$-closed) subset of $(Y, \sigma)$, then $f^{-1}(A)$ is $\omega$-open (or $\omega$-closed) in $(X, \tau)$.

Proof: Let $A$ be an $\omega$-open set in $(Y, \sigma)$ and $F$ be any semi-closed set in $(X, \tau)$ such that $F \subseteq f^{-1}(A)$. Then $f(F) \subseteq A$. By hypothesis, $f(F)$ is semi-closed and $A$ is $\omega$-open in $(Y, \sigma)$. Therefore, $f(F) \subseteq \text{int}(A)$ by Theorem 2.4.6 and so $F \subseteq f^{-1}(\text{int}(A))$. Since $f$ is $\omega$-continuous and $\text{int}(A)$ is open in $(Y, \sigma)$,
f_1(int(A)) is \( \omega \)-open in \((X, \tau)\). Thus \( F \subseteq \text{int}(f^{-1}(\text{int}(A))) \subseteq \text{int}(f^{-1}(A)) \). i.e., \( F \subseteq \text{int}(f^{-1}(A)) \) and by Theorem 2.4.6, \( f^{-1}(A) \) is \( \omega \)-open in \((X, \tau)\).

By taking complements, we can show that if \( A \) is \( \omega \)-closed in \((Y, \sigma)\), \( f^{-1}(A) \) is \( \omega \)-closed in \((X, \tau)\).

**Corollary 3.2.31** If \( f : (X, \tau) \to (Y, \sigma) \) is continuous and pre-semi-closed and if \( B \) is a \( \omega \)-closed (or \( \omega \)-open) subset of \((Y, \sigma)\), then \( f^{-1}(B) \) is \( \omega \)-closed (or \( \omega \)-open) in \((X, \tau)\).

**Proof:** Follows from Proposition 3.2.3 and Theorem 3.2.30.

**Corollary 3.2.32** Let \((X, \tau), (Y, \sigma)\) and \((Z, \eta)\) be any three topological spaces. If \( f : (X, \tau) \to (Y, \sigma) \) is \( \omega \)-continuous and pre-semi-closed and \( g : (Y, \sigma) \to (Z, \eta) \) is \( \omega \)-continuous, then their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is \( \omega \)-continuous.

**Proof:** Let \( F \) be any closed set in \((Z, \eta)\). Since \( g : (Y, \sigma) \to (Z, \eta) \) is \( \omega \)-continuous, \( g^{-1}(F) \) is \( \omega \)-closed in \((Y, \sigma)\). Since \( f : (X, \tau) \to (Y, \sigma) \) is \( \omega \)-continuous and pre-semi-closed, by Theorem 3.2.30, \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is \( \omega \)-closed in \((X, \tau)\) and so \( g \circ f \) is \( \omega \)-continuous.

**Definition 3.2.33** Let \( x \) be a point of \((X, \tau)\) and \( W \) be a subset of \((X, \tau)\). Then \( W \) is called an \( \omega \)-neighborhood of \( x \) in \((X, \tau)\) if there exists an \( \omega \)-open set \( U \) of \((X, \tau)\) such that \( x \in U \subseteq W \).

**Proposition 3.2.34** Let \( A \) be a subset of \((X, \tau)\). Then \( x \in \omega \text{-cl}(A) \) if and only if for any \( \omega \)-neighborhood \( W_x \) of \( x \) in \((X, \tau)\), \( A \cap W_x \neq \emptyset \).

**Proof:** Necessity: Assume \( x \in \omega \text{-cl}(A) \). Suppose that there is an \( \omega \)-neighborhood \( W \) of the point \( x \) in \((X, \tau)\) such that \( W \cap A = \emptyset \). Since \( W \) is a \( \omega \)-neighborhood of \( x \) in \((X, \tau)\), by Definition 3.2.33, there exists an \( \omega \)-open set \( U_x \) such that \( x \in U_x \subseteq W \). Therefore, we have \( U_x \cap A = \emptyset \) and so \( A \subseteq (U_x)^c \). Since \((U_x)^c\) is an \( \omega \)-closed set containing \( A \), we have by Definition 2.3.1, \( \omega \text{-cl}(A) \subseteq (U_x)^c \) and therefore \( x \notin \omega \text{-cl}(A) \), which is a contradiction.
Sufficiency: Assume for each $\omega$-neighborhood $W_x$ of $x$ in $(X, \tau)$, $A \cap W_x \neq \emptyset$. Suppose that $x \notin \omega\text{-cl}(A)$. Then by Definition 2.3.1, there exists an $\omega$-closed set $F$ of $(X, \tau)$ such that $A \subseteq F$ and $x \notin F$. Thus $x \in F^c$ and $F^c$ is $\omega$-open in $(X, \tau)$ and hence $F^c$ is a $\omega$-neighborhood of $x$ in $(X, \tau)$. But $A \cap F^c = \emptyset$, which is a contradiction.

In the next theorem we explore certain characterizations of $\omega$-continuous functions.

**Theorem 3.2.35** Let $f: (X, \tau) \to (Y, \sigma)$ be a map from a topological space $(X, \tau)$ into a topological space $(Y, \sigma)$. Then the following statements are equivalent:

a) The function $f$ is $\omega$-continuous.

b) The inverse of each open set is $\omega$-open.

c) For each point $x$ in $(X, \tau)$ and each open set $V$ in $(Y, \sigma)$ with $f(x) \in V$, there is an $\omega$-open set $U$ in $(X, \tau)$ such that $x \in U$, $f(U) \subseteq V$.

d) The inverse of each closed set is $\omega$-closed.

e) For each $x$ in $(X, \tau)$, the inverse of every neighborhood of $f(x)$ is an $\omega$-neighborhood of $x$.

f) For each $x$ in $(X, \tau)$ and each neighborhood $N$ of $f(x)$, there is an $\omega$-neighborhood $W$ of $x$ such that $f(W) \subseteq N$.

g) For each subset $A$ of $(X, \tau)$, $f(\omega\text{-cl}(A)) \subseteq \text{cl}(f(A))$.

h) For each subset $B$ of $(Y, \sigma)$, $\omega\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$.

**Proof:** a) $\iff$ b): This follows from Proposition 3.2.15.

(a) $\iff$ c): Suppose that (c) holds and let $V$ be an open set in $(Y, \sigma)$ and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and thus there exists an $\omega$-open set $U_x$ such that $x \in U_x$ and $f(U_x) \subseteq V$. Now, $x \in U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \bigcup_{v \in f^{-1}(V)} U_x$. Then by Proposition 2.4.4, $f^{-1}(V)$ is $\omega$-open in $(X, \tau)$ and therefore $f$ is $\omega$-continuous.
Conversely, suppose that (a) holds and let \( f(x) \in V \). Then \( x \in f^{-1}(V) \in \tau^\omega \), since \( f \) is \( \omega \)-continuous. Let \( U = f^{-1}(V) \). Then \( x \in U \) and \( f(U) \subseteq V \).

b) \( \iff \) d) : This result follows from the fact that if \( A \) is a subset of \((Y, \sigma)\), then \( f^{-1}(A^c) = (f^{-1}(A))^c \).

b) \( \Rightarrow \) e) : For \( x \) in \((X, \tau)\), let \( N \) be a neighborhood of \( f(x) \). Then there exists an open set \( U \) in \((Y, \sigma)\) such that \( f(x) \in U \subseteq N \). Consequently, \( f^{-1}(U) \) is an \( \omega \)-open set in \((X, \tau)\) and \( x \in f^{-1}(U) \subseteq f^{-1}(N) \). Thus \( f^{-1}(N) \) is an \( \omega \)-neighborhood of \( x \).

e) \( \Rightarrow \) f) : Let \( x \in X \) and let \( N \) be a neighborhood of \( f(x) \). Then by assumption, \( W = f^{-1}(N) \) is an \( \omega \)-neighborhood of \( x \) and \( f(W) = f(f^{-1}(N)) \subseteq N \).

f) \( \Rightarrow \) c) : For \( x \) in \((X, \tau)\), let \( V \) be an open set containing \( f(x) \). Then \( V \) is a neighborhood of \( f(x) \). So by assumption, there exists an \( \omega \)-neighborhood \( W \) of \( x \) such that \( f(W) \subseteq V \). Hence there exists an \( \omega \)-open set \( U \) in \((X, \tau)\) such that \( x \in U \subseteq W \) and so \( f(U) \subseteq f(W) \subseteq V \).

g) \( \iff \) d) : Suppose that (d) holds and let \( A \) be a subset of \((X, \tau)\). Since \( A \subseteq f^{-1}(f(A)) \), we have \( A \subseteq f^{-1}(\text{cl}(f(A))) \). Since \( \text{cl}(f(A)) \) is a closed set in \((Y, \sigma)\), by assumption \( f^{-1}(\text{cl}(f(A))) \) is an \( \omega \)-closed set containing \( A \). Consequently, \( \omega-\text{cl}(A) \subseteq f^{-1}(\text{cl}(f(A))) \). Thus \( f(\omega-\text{cl}(A)) \subseteq f(f^{-1}(\text{cl}(f(A)))) \subseteq \text{cl}(f(A)) \).

Conversely, suppose that (g) holds for any subset \( A \) of \((X, \tau)\). Let \( F \) be a closed subset of \((Y, \sigma)\). Then by assumption, \( f(\omega-\text{cl}(f^{-1}(F))) \subseteq \text{cl}(f(f^{-1}(F))) \subseteq \text{cl}(F) = F \). i.e., \( \omega-\text{cl}(f^{-1}(F)) \subseteq f^{-1}(F) \) and so \( f^{-1}(F) \) is \( \omega \)-closed.

g) \( \iff \) h) : Suppose that (g) holds and \( B \) be any subset of \((Y, \sigma)\). Then replacing \( A \) by \( f^{-1}(B) \) in g), we obtain \( f(\omega-\text{cl}(f^{-1}(B))) \subseteq \text{cl}(f(f^{-1}(B))) \subseteq \text{cl}(B) \).

Conversely, suppose that (h) holds. Let \( B = f(A) \) where \( A \) is a subset of \((X, \tau)\). Then we have, \( \omega-\text{cl}(A) \subseteq \omega-\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(f(A))) \) and so \( f(\omega-\text{cl}(A)) \subseteq \text{cl}(f(A)) \).
This completes the proof of the theorem.

**Definition 3.2.36** Let $A$ be a subset of $(X, \tau)$. Then a mapping $r : (X, \tau) \to (A, \tau_A)$ is called an $\omega$-continuous retraction if $r$ is $\omega$-continuous and the restriction $r_A$ is the identity mapping on $A$.

**Theorem 3.2.37** Let $A$ be a subset of $(X, \tau)$ and $r : (X, \tau) \to (A, \tau_A)$ be an $\omega$-continuous retraction. If $(X, \tau)$ is Hausdorff, then $A$ is an $\omega$-closed set of $(X, \tau)$.

**Proof**: Suppose that $A$ is not $\omega$-closed. Then by Proposition 2.3.9, there exists a point $x$ in $(X, \tau)$ such that $x \in \omega\text{-cl}(A)$ but $x \notin A$. Since $r$ is an $\omega$-continuous retraction, we have $r(x) \neq x$ and since $(X, \tau)$ is Hausdorff, there exists disjoint open sets $U$ and $V$ in $(X, \tau)$ such that $x \in U$ and $r(x) \in V$. Let $W$ be an arbitrary $\omega$-neighborhood of $x$. Then $U \cap W$ is an $\omega$-neighborhood of $x$. Since $x \in \omega\text{-cl}(A)$, by Proposition 3.2.34, we have $(U \cap W) \cap A \neq \emptyset$ and so there exists a point $y$ in $(U \cap W) \cap A$. Since $y \in A$, we have $r(y) = y \in U$ and hence $r(y) \notin V$. Since $y \in W$, $r(W) \subseteq V$, which is a contradiction to $\omega$-continuity of $r$. Therefore $A$ is an $\omega$-closed subset of $(X, \tau)$.

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**3.3. $\omega$-irresolute maps, strongly $\omega$-continuous maps and perfectly $\omega$-continuous maps**

Irresolute functions are introduced and studied by Crossely and Hildebrand [19]. Recently Sundaram [112] and Devi et al [22, 23] have investigated gc-irresolute, $\alpha g$-irresolute and gs-irresolute functions. In this section we introduce the concepts of $\omega$-irresolute map, strongly $\omega$-continuous map and perfectly $\omega$-continuous map in topological spaces and investigate some of their properties.
**Definition 3.3.1** A map \( f : (X, \tau) \to (Y, \sigma) \) is called an \( \omega \)-irresolute map if the inverse image of every \( \omega \)-closed set in \( (Y, \sigma) \) is \( \omega \)-closed in \( (X, \tau) \).

**Remark 3.3.2** The following examples show that the notion of irresolute maps and \( \omega \)-irresolute maps are independent.

**Example 3.3.3** Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{b, c\}, Y\} \). Then the identity map on \( X \) is \( \omega \)-irresolute but it is not irresolute.

**Example 3.3.4** Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{b, c\}, X\} \). Then the map \( f : (X, \tau) \to (Y, \sigma) \) defined by \( f(a) = b, f(b) = a \) and \( f(c) = c \) is irresolute but not \( \omega \)-irresolute.

**Proposition 3.3.5** A map \( f : (X, x) \to (Y, a) \) is \( \omega \)-irresolute if and only if the inverse image of every \( \omega \)-open set in \( (Y, a) \) is \( \omega \)-open in \( (X, x) \).

**Proof:** Similar to Proposition 3.2.15.

**Proposition 3.3.6** If a map \( f : (X, \tau) \to (Y, \sigma) \) is \( \omega \)-irresolute then it is \( \omega \)-continuous but not conversely.

**Proof:** Follows from Proposition 2.2.2.

**Example 3.3.7** Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, \{a\}, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{a, b\}, Y\} \). Then the identity map on \( X \) is \( \omega \)-continuous but not \( \omega \)-irresolute as the inverse image of the \( \omega \)-closed set \( \{a, c\} \) in \( (Y, \sigma) \) is \( \{a, c\} \) which is not \( \omega \)-closed in \( (X, \tau) \).

**Proposition 3.3.8** Let \( (X, \tau) \) be any topological space, \( (Y, \sigma) \) be a \( T_\omega \) space and \( f : (X, \tau) \to (Y, \sigma) \) be a map. Then the following are equivalent:

i) \( f \) is \( \omega \)-irresolute, ii) \( f \) is \( \omega \)-continuous.

**Proof:** i) \( \Rightarrow \) ii): Follows from Proposition 3.3.6.

ii) \( \Rightarrow \) i): Let \( F \) be an \( \omega \)-closed set in \( (Y, \sigma) \). Since \( (Y, \sigma) \) is a \( T_\omega \) space, \( F \) is a closed set in \( (Y, \sigma) \) and by hypothesis, \( f^{-1}(F) \) is \( \omega \)-closed in \( (X, \tau) \). Therefore \( f \) is \( \omega \)-irresolute.
Proposition 3.3.9 If $f: (X, \tau) \to (Y, \sigma)$ is $\omega$-irresolute and $H$ is an $\omega$-closed subset of $(X, \tau)$, then the restriction $f_H: (H, \tau_H) \to (Y, \sigma)$ is $\omega$-irresolute.

Proof: Let $F$ be any $\omega$-closed subset of $(Y, \sigma)$. Since $f$ is $\omega$-irresolute, $f^{-1}(F)$ is $\omega$-closed in $(X, \tau)$. Let $f^{-1}(F) \cap H = H_1$. Then $H_1$ is $\omega$-closed in $(H, \tau_H)$ as in the proof of Proposition 3.2.22. But $(f_H)^{-1}(F) = f^{-1}(F) \cap H = H_1$ and so $f_H$ is also $\omega$-irresolute.

Proposition 3.3.10 If $f: (X, \tau) \to (Y, \sigma)$ is bijective, pre-semi-open and $\omega$-continuous then $f$ is $\omega$-irresolute.

Proof: Let $A$ be $\omega$-closed set in $(Y, \sigma)$. Let $U$ be any semi-open set in $(X, \tau)$ such that $f^{-1}(A) \subseteq U$. Then $A \subseteq f(U)$. Since $A$ is $\omega$-closed and $f(U)$ is semi-open in $(Y, \sigma)$, $\text{cl}(A) \subseteq f(U)$ holds and hence $f^{-1}(\text{cl}(A)) \subseteq U$. Since $f$ is $\omega$-continuous and $\text{cl}(A)$ is closed in $(Y, \sigma)$, $\text{cl}(f^{-1}(\text{cl}(A))) \subseteq U$ and so $\text{cl}(f^{-1}(A)) \subseteq U$. Therefore, $f^{-1}(A)$ is $\omega$-closed in $(X, \tau)$ and hence $f$ is $\omega$-irresolute.

Remark 3.3.11 The following examples show that no assumption of Proposition 3.3.10 can be removed.

Example 3.3.12 Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(c) = b$ and $f(b) = c$. Then $f$ is $\omega$-continuous and pre-semi-open but it is not bijective and so $f$ is not $\omega$-irresolute.

Example 3.3.13 The identity map in Example 3.3.7 is $\omega$-continuous and bijective but not pre-semi-open and so $f$ is not $\omega$-irresolute.

Example 3.3.14 Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Define $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then $f$ is bijective and pre-semi-open but not $\omega$-continuous and so $f$ is not $\omega$-irresolute.
Proposition 3.3.15 If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is bijective, closed and irresolute then the inverse map \( f^{-1} : (Y, \sigma) \rightarrow (X, \tau) \) is \( \omega \)-irresolute.

**Proof:** Let \( A \) be \( \omega \)-closed in \( (X, \tau) \). Let \( (f^{-1})^{-1}(A) = f(A) \subseteq U \) where \( U \) is semi-open in \( (Y, \sigma) \). Then \( A \subseteq f^{-1}(U) \) holds. Since \( f^{-1}(U) \) is semi-open in \( (X, \tau) \) and \( A \) is \( \omega \)-closed in \( (X, \tau) \), \( cl(A) \subseteq f^{-1}(U) \) and hence \( f(cl(A)) \subseteq U \).

Since \( f \) is closed and \( cl(A) \) is closed in \( (X, \tau) \), \( f(cl(A)) \) is closed in \( (Y, \sigma) \) and so \( f(cl(A)) \) is \( \omega \)-closed in \( (Y, \sigma) \). Therefore \( cl(f(cl(A))) \subseteq U \) and hence \( cl(f(A)) \subseteq U \). Thus \( f(A) \) is \( \omega \)-closed in \( (Y, \sigma) \) and so \( f^{-1} \) is \( \omega \)-irresolute.

We introduce the following definition.

**Definition 3.3.16** A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called strongly \( \omega \)-continuous if the inverse image of every \( \omega \)-open set in \( (Y, \sigma) \) is open in \( (X, \tau) \).

Proposition 3.3.17 If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is strongly \( \omega \)-continuous, then it is continuous but not conversely.

**Proof:** Follows from Propositions 2.4.2.

**Example 3.3.18** Let \( X = \{a, b, c\} \), \( Y = \{p, q, r\} \), \( \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{p, q\}, Y\} \). Define a map \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = f(b) = p \) and \( f(c) = q \). Then \( f \) is continuous. But \( f \) is not strongly \( \omega \)-continuous, since for the \( \omega \)-open set \( U = \{p\} \) in \( (Y, \sigma) \), \( f^{-1}(U) = \{a, b\} \) is not open in \( (X, \tau) \).

**Proposition 3.3.19** Let \( (X, \tau) \) be any topological space and \( (Y, \sigma) \) be a \( T_{\infty} \) space and \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a map. Then the following are equivalent:

i) \( f \) is strongly \( \omega \)-continuous, ii) \( f \) is continuous.

**Proof:** i) \( \Rightarrow \) ii): Follows from Proposition 3.3.17.

ii) \( \Rightarrow \) i): Let \( U \) be any \( \omega \)-open set in \( (Y, \sigma) \). Since \( (Y, \sigma) \) is a \( T_{\infty} \) space, \( U \) is open in \( (Y, \sigma) \) and since \( f \) is continuous, we have \( f^{-1}(U) \) is open in \( (X, \tau) \). Therefore \( f \) is strongly \( \omega \)-continuous.
Proposition 3.3.20 If \( f : (X, \tau) \to (Y, \sigma) \) is strongly continuous then it is strongly \( \omega \)-continuous but not conversely.
Proof is easy consequences from definitions.

Example 3.3.21 Let \( X = \{a, b, c\} = Y \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a, b\}, Y\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity map. Then \( f \) is strongly \( \omega \)-continuous. But \( f \) is not strongly continuous, since for the subset \( A = \{a\} \) of \( (Y, \sigma) \), \( f^{-1}(A) = \{a\} \) is open in \( (X, \tau) \) but not closed in \( (X, \tau) \).

Definition 3.3.22 A topological space \((X, \tau)\) is called an \( \omega \)-space if every subset in it is \( \omega \)-closed, i.e., \( \tau^\omega = \mathcal{P}(X) \).

For example, the topological space \((X, \tau)\) in Example 3.2.2 is an \( \omega \)-space, because \( \tau^\omega = \mathcal{P}(X) \).

Proposition 3.3.23 Let \((X, \tau)\) be a discrete topological space, \((Y, \sigma)\) be an \( \omega \)-space and \( f : (X, \tau) \to (Y, \sigma) \) be a map. Then the following are equivalent:

i) \( f \) is strongly continuous, ii) \( f \) is strongly \( \omega \)-continuous.

Proof: i) \( \Rightarrow \) ii): Follows from Proposition 3.3.20.

ii) \( \Rightarrow \) i): Let \( U \) be any \( \omega \)-open set in \((Y, \sigma)\). Since \((Y, \sigma)\) is an \( \omega \)-space, \( U \) is an \( \omega \)-open subset of \((Y, \sigma)\) and by hypothesis, \( f^{-1}(U) \) is open in \((X, \tau)\). But \((X, \tau)\) is a discrete topological space and so \( f^{-1}(U) \) is also closed in \((X, \tau)\), i.e., \( f^{-1}(U) \) is both open and closed in \((X, \tau)\) and so \( f \) is strongly continuous.

Proposition 3.3.24 Let \( f : (X, \tau) \to (Y, \sigma) \) be a map and both \((X, \tau)\) and \((Y, \sigma)\) be \( T_\omega \)-spaces. Then the following are equivalent:

i) \( f \) is strongly \( \omega \)-continuous, ii) \( f \) is continuous, iii) \( f \) is \( \omega \)-irresolute and iv) \( f \) is \( \omega \)-continuous.

Proof: Follows from Propositions 3.3.8 and 3.3.19.
**Proposition 3.3.25** A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is strongly \( \omega \)-continuous if and only if the inverse image of every \( \omega \)-closed set in \((Y, \sigma)\) is closed in \((X, \tau)\).

**Proof:** Similar to Proposition 3.2.15.

**Proposition 3.3.26** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) are strongly \( \omega \)-continuous, then their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is also strongly \( \omega \)-continuous.

**Proof:** Let \( U \) be an \( \omega \)-open set in \((Z, \eta)\). Since \( g \) is strongly \( \omega \)-continuous, \( g^{-1}(U) \) is open in \((Y, \sigma)\). Since \( g^{-1}(U) \) is open, it is \( \omega \)-open in \((Y, \sigma)\). As \( f \) is also strongly \( \omega \)-continuous, \( f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \) is open in \((X, \tau)\) and so \( g \circ f \) is strongly \( \omega \)-continuous.

**Proposition 3.3.27** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be any two maps. Then their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is

i) strongly \( \omega \)-continuous if \( g \) is strongly \( \omega \)-continuous and \( f \) is continuous,

ii) \( \omega \)-irresolute if \( g \) is strongly \( \omega \)-continuous and \( f \) is \( \omega \)-continuous (or \( f \) is \( \omega \)-irresolute),

iii) strongly \( \omega \)-continuous if \( g \) is strongly continuous and \( f \) is irresolute,

iv) continuous if \( g \) is \( \omega \)-continuous and \( f \) is strongly \( \omega \)-continuous.

**Proof:** Similar to Proposition 3.3.26.

**Proposition 3.3.28** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is strongly \( \omega \)-continuous and \( H \) is any subset of \((X, \tau)\), then the restriction \( f_H : (H, \tau_H) \rightarrow (Y, \sigma) \) is also strongly \( \omega \)-continuous.

**Proof:** Similar to Proposition 3.3.9.

We next introduce perfectly \( \omega \)-continuous maps in topological spaces.
**Definition 3.3.29** A map \( f: (X, \tau) \to (Y, \sigma) \) is called perfectly \( \omega \)-continuous if the inverse image of every \( \omega \)-open set in \((Y, \sigma)\) is both open and closed in \((X, \tau)\).

**Proposition 3.3.30** If \( f: (X, \tau) \to (Y, \sigma) \) is perfectly \( \omega \)-continuous then it is strongly \( \omega \)-continuous but not conversely.

**Proof:** Since \( f: (X, \tau) \to (Y, \sigma) \) is perfectly \( \omega \)-continuous, \( f^{-1}(U) \) is both open and closed in \((X, \tau)\) for every \( \omega \)-open set \( U \) in \((Y, \sigma)\). Therefore \( f \) is strongly \( \omega \)-continuous.

**Example 3.3.31** Let \((X, \tau)\) and \((Y, \sigma)\) be the topological spaces of Examples 3.3.21 and 3.3.18 respectively. Define a map \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = p, \ f(b) = q \) and \( f(c) = r \). Then \( f \) is strongly \( \omega \)-continuous. But \( f \) is not perfectly \( \omega \)-continuous, since for the \( \omega \)-open set \( G = \{q\} \) in \((Y, \sigma)\), \( f^{-1}(G) = \{b\} \) is open but not closed in \((X, \tau)\).

**Proposition 3.3.32** If \( f: (X, \tau) \to (Y, \sigma) \) is strongly continuous then it is perfectly \( \omega \)-continuous but not conversely.

**Proof:** Since \( f: (X, \tau) \to (Y, \sigma) \) is strongly continuous, \( f^{-1}(U) \) is both open and closed in \((X, \tau)\), for every \( \omega \)-open set \( U \) in \((Y, \sigma)\). Therefore \( f \) is perfectly \( \omega \)-continuous.

**Example 3.3.33** Let \( X = Y = \{a, b, c\}, \ \tau = \{\phi, \{a\}, \{b, c\}, X\}, \ \sigma = \{\phi, \{a\}, Y\} \) and \( f: (X, \tau) \to (Y, \sigma) \) be the identity map on \( X \). Then \( f \) is perfectly \( \omega \)-continuous but not strongly continuous.

**Remark 3.3.34** From the above observations we have the following implications:

- Perfectly \( \omega \)-continuity \( \implies \) Strongly \( \omega \)-continuity
- Strong continuity \( \implies \) Continuity

\[ \begin{array}{ccc}
\text{Perfectly } \omega \text{-continuity} & \longrightarrow & \text{Strongly } \omega \text{-continuity} \\
\uparrow & & \downarrow \\
\text{Strong continuity} & \longrightarrow & \text{Continuity}
\end{array} \]
**Proposition 3.3.35** Let \((X, \tau)\) be a discrete topological space, \((Y, \sigma)\) be any topological space and \(f: (X, \tau) \to (Y, \sigma)\) be a map. Then the following are equivalent:

i) \(f\) is perfectly \(\omega\)-continuous, ii) \(f\) is strongly \(\omega\)-continuous.

**Proof:** i) \(\Rightarrow\) ii): Follows from Proposition 3.3.30.

ii) \(\Rightarrow\) i): Let \(U\) be any \(\omega\)-open set in \((Y, \sigma)\). By hypothesis \(f^{-1}(U)\) is open in \((X, \tau)\). Since \((X, \tau)\) is a discrete space, \(f^{-1}(U)\) is also closed in \((X, \tau)\). i.e., \(f^{-1}(U)\) is both open and closed in \((X, \tau)\) and so \(f\) is perfectly \(\omega\)-continuous.

**Proposition 3.3.36** Let \((X, \tau)\) be a discrete topological space, \((Y, \sigma)\) be an \(\omega\)-space and \(f: (X, \tau) \to (Y, \sigma)\) be a map. Then the following are equivalent:

i) \(f\) is strongly continuous, ii) \(f\) is strongly \(\omega\)-continuous, iii) \(f\) is perfectly \(\omega\)-continuous.

**Proof:** Follows from Propositions 3.3.23 and 3.3.35.

**Proposition 3.3.37** A map \(f: (X, \tau) \to (Y, \sigma)\) is perfectly \(\omega\)-continuous if and only if the inverse image of every \(\omega\)-closed set in \((Y, \sigma)\) is both open and closed in \((X, \tau)\).

**Proof:** Similar to Proposition 3.2.15.

**Proposition 3.3.38** If \(f: (X, \tau) \to (Y, \sigma)\) and \(g: (Y, \sigma) \to (Z, \eta)\) are perfectly \(\omega\)-continuous, then their composition \(g \circ f: (X, \tau) \to (Z, \eta)\) is also perfectly \(\omega\)-continuous.

**Proof:** Similar to Proposition 3.3.26.

**Proposition 3.3.39** Let \(f: (X, \tau) \to (Y, \sigma)\) and \(g: (Y, \sigma) \to (Z, \eta)\) be any two maps. Then their composition \(g \circ f: (X, \tau) \to (Z, \eta)\) is

i) \(\omega\)-continuous if \(g\) is strongly continuous and \(f\) is \(\omega\)-continuous,
ii) \(\omega\)-irresolute if \(g\) is perfectly \(\omega\)-continuous and \(f\) is \(\omega\)-continuous (or \(f\) is \(\omega\)-irresolute),

iii) strongly \(\omega\)-continuous if \(g\) is perfectly \(\omega\)-continuous and \(f\) is continuous (or \(f\) is strongly \(\omega\)-continuous),

iv) perfectly \(\omega\)-continuous if \(g\) is strongly continuous and \(f\) is perfectly \(\omega\)-continuous.

**Proof:** Similar to Proposition 3.3.26.

**Propositions 3.3.40** If \(f : (X, \tau) \to (Y, \sigma)\) is perfectly \(\omega\)-continuous and \(H\) is any subset of \((X, \tau)\), then the restriction \(f_H : (H, \tau_H) \to (Y, \sigma)\) is also perfectly \(\omega\)-continuous.

**Proof:** Similar to Proposition 3.3.9.

3.4. Applications

In this section by using \(\omega\)-continuity we obtain a decomposition of continuity in topological spaces. As another application of \(\omega\)-continuous maps in topological spaces, we construct some new sandwich type near-rings.

The decomposition of continuity is one of the many problems in general topology. Tong [117] introduced the notions of A-sets and A-continuity and established a decomposition of continuity. Also tong [118] introduced the notions of B-sets and B-continuity and used them to obtain another decomposition of continuity and Ganster and Reilly [46] have improved Tong's decomposition result. Przemiński [101] obtained some decomposition of continuity. Hatir, Noiri and Yuksel [52] also obtained a
decomposition of continuity. Recently Dontchev, Przemski [36] and Rajamani [103] obtained some more decompositions of continuity.

To obtain our decomposition of continuity, we first introduce the notion of slc*-set and slc*-continuous mapping in topological spaces and we prove that a map is continuous if and only if it is both ω-continuous and slc*-continuous.

**Definition 3.4.1** [48] A subset A of (X, τ) is called semi-locally closed (briefly slc) if A = U ∩ F where U is semi-open and f is semi-closed in (X, τ).

We introduce the following definition.

**Definition 3.4.2** A subset A of (X, τ) is called slc* if A = U ∩ F where U is semi-open and F is closed in (X, τ).

**Example 3.4.3** Let (X, τ) be the topological space of Example 3.2.6. Then the set A = {b} is a slc* set in (X, τ).

**Remark 3.4.4** Every closed set is slc* but not conversely. In Example 3.2.6, the set A = {a, b} is a slc* set but not a closed set in (X, τ).

**Remark 3.4.5** ω-closed sets and slc* sets are independent. Consider the topological space (X, τ) of Example 3.2.2. Then the set A = {a, b} is an ω-closed set but not a slc* set in (X, τ). Let (X, τ) be the topological space of Example 3.2.6. Then the set B = {a, c} is a slc*-set but not an ω-closed set in (X, τ).

**Proposition 3.4.6** Let (X, τ) be a topological space. Then a subset A of (X, τ) is closed if and only if it is both ω-closed and slc* set.

**Proof**: Necessity is trivial.

To prove the sufficiency, assume that A is both ω-closed and slc* set. Then A = U ∩ F where U is semi-open and F is closed in (X, τ). Therefore A ⊆ U
and $A \subseteq F$ and so by hypothesis, $\text{cl}(A) \subseteq U$ and $\text{cl}(A) \subseteq F$. Thus $\text{cl}(A) \subseteq U \cap F = A$ and hence $\text{cl}(A) = A$ i.e., $A$ is closed in $(X, \tau)$.

**Definition 3.4.7** A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be slc*-continuous if for each closed set $F$ in $(Y, \sigma)$, $f^{-1}(F)$ is a slc* set in $(X, \tau)$.

**Example 3.4.8** Let $(X, \tau)$ be the topological space of Example 3.2.6 and $Y = \{a, b, c\}, \sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map on $X$. Then $f$ is a slc*-continuous map.

**Remark 3.4.9** From the definitions it is clear that every continuous map is slc*-continuous. But the converse is not true. The map $f$ in Example 3.4.8 is slc*-continuous but not continuous, since for the closed set $\{a\}$ in $(Y, \sigma)$, $f^{-1}(\{a\}) = \{a\}$, which is not closed in $(X, \tau)$.

**Remark 3.4.10** $\omega$-continuity and slc*-continuity are independent. The function $f$ in Example 3.2.2 is $\omega$-continuous but not slc*-continuous and the function $f$ in Example 3.2.6 is slc*-continuous but not $\omega$-continuous.

We have the following decomposition for continuity.

**Theorem 3.4.11** A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous if and only if it is both $\omega$-continuous and slc*-continuous.

**Proof:** Assume $f$ is continuous. Then by Proposition 3.2.3 and Remark 3.4.9, $f$ is both $\omega$-continuous and slc*-continuous.

Conversely, assume that $f$ is both $\omega$-continuous and slc*-continuous. Let $F$ be a closed subset of $(Y, \sigma)$. Then $f^{-1}(F)$ is both $\omega$-closed and slc*-set. As in Proposition 3.4.6, we prove that $f^{-1}(F)$ is a closed set in $(X, \tau)$ and so $f$ is continuous.

Recently Veera Kumar [120, 122] constructed some sandwich type near-rings using the concept of generalized continuities in topological spaces. As an application of $\omega$-continuous maps, we construct two new sandwich near-rings.
Near-rings are generalized rings. A near-ring is a ring \( (N, +, \circ) \), where + is not necessarily abelian and with only one distributive law holds. Near-rings arise in a natural way: Consider the set \( N(f) = \{ f : (G, +) \to (G, +) \} \), where \((G, +)\) is a group. i.e., \( N(f) \) is the set of all mappings from a group \((G, +)\) into itself. Define addition + as pointwise addition and \( \circ \) as composition. Then \((N(f), +, \circ)\) is a near-ring. Even if G is abelian, only one distributive law is fulfilled: \((f_1 + f_2) \circ f_3 = f_1 \circ f_3 + f_2 \circ f_3\) holds by definition of \(f_1 + f_2\), while for \(f_1 \circ (f_2 + f_3) = f_1 \circ f_2 + f_1 \circ f_3\) we would have to assume that \(f_1\) is a homomorphism.

First we recall the following definitions and results which are used in our present study.

**Definition 3.4.12[120]** A topological group is a triple \((X, \cdot, \tau)\) such that \((X, \cdot)\) is a group and \((X, \tau)\) is a topological space such that the functions \(x \to x^{-1}\) and \((x, y) \to x \cdot y\) are continuous.

**Definition 3.4.13[120]** A right near-ring is a triple \((N, +, \cdot)\), where + and \(\cdot\) are binary operations on the set \(N\), where \((N, +)\) is a group (not necessarily abelian) and \((N, \cdot)\) is a semi-group and satisfies the right distributive law:
\[(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3\]
for all \(n_1, n_2, n_3 \in N\).

A subgroup \((M, +)\) of a near-ring \((N, +, \cdot)\) is called a subnear-ring of \(N\) if \(M \cdot M \subseteq M\) and as in [99], we write \(M \leq N\).

**Definition 3.4.14[120]** A mapping \(f : (N_1, \oplus, \otimes) \to (N_2, +, \cdot)\) is called a near-ring homomorphism if \(f(m \oplus n) = f(m) + f(n)\) and \(f(m \otimes n) = f(m) \cdot f(n)\), for all \(m, n \in N_1\).

If \(f\) is bijective, then \(f\) is called a near-ring isomorphism and we say that \(N_1\) and \(N_2\) are isomorphic. Symbolically we write \(N_1 \cong N_2\).

**Theorem 3.4.15[120]** Let \((X, \cdot, \tau)\) be any topological group and \(A \subseteq X\). Then for any \(b \in X\), \(b \cdot A\) and \(A \cdot b\) are open if and only if \(A\) is open.
Corollary 3.4.16[120] For each point x in a topological group \((X, *, t)\), the left and right translations \(L_a\) and \(R_a\) defined by \(L_a(x) = a * x\) and \(R_a(x) = x * a\) for any fixed \(a \in X\) are homeomorphisms.

Theorem 3.4.17 Let \((X, *, t)\) be any topological group and \(A \subseteq X\).
Then for any \(b \in X\), \(b * A\) and \(A * b\) are closed if and only if \(A\) is closed.

Proof: Follows from Theorem 3.4.15.

Now we construct some new sandwich type right near-rings using Theorem 2.2.20 and \(\omega\)-continuous maps.

Let \((X, \tau)\) be a topological space and \((G, * , \sigma)\) be a topological group such that \((X, \tau) = (G, \sigma)\) and let \(C(X, G) = \{f : (X, \tau) \to (G, \sigma)\text{ is continuous}\}\). Then \((C(X, G), \oplus, \cdot)\), where \((f \oplus g)(x) = f(x) \cdot g(x)\) for any \(f, g \in C(X, G)\), for any \(x \in X\) and \(\cdot\) is the composition of maps is a right near-ring.

Considering the case \((X, \tau) \neq (G, \sigma)\) and fixing a continuous map \(\phi : (G, \sigma) \to (X, \tau)\) Magill Jr., [60] defined a binary operation \(\circ_\phi\) on \(C(X, G)\) as \(f \circ_\phi g = f \circ \phi \circ g\). With such a \(\phi\), \((C(X, G), \oplus, \circ_\phi)\) is a near-ring. Such near-rings are called sandwich near-rings.

Let \(G_{\omega}^X = \{f : (X, \tau) \to (G, \sigma)\text{ is }\omega\text{-continuous}\}\). Then \(G_{\omega}^X\) is not closed with respect to the binary operation \(\circ\), the composition of mappings on \(G_{\omega}^X\). This is shown in Example 3.2.18.

We shall make \(G_{\omega}^X\) as a sandwich type near-ring as follows:

Theorem 3.4.18 Let \((X, \tau)\) be a topological space, \((G, +, \sigma)\) be a topological group with \((X, \tau) \neq (G, \sigma)\) and \((G, \sigma)\) be a \(T_\omega\) space. Then \(G_{\omega}^X\) can be made into a sandwich type near-ring.

Proof: Define \(\oplus : G_{\omega}^X \times G_{\omega}^X \to G_{\omega}^X\) by \(f \oplus g = f + g\) for all \(f, g \in G_{\omega}^X\), where \((f + g)(x) = f(x) + g(x)\) for all \(x \in X\) and let \(F\) be a closed subset of \((G, \sigma)\). Then it is easy to see that \((f \oplus g)^{-1}(F) = f^{-1}(F - g(x)) \cap \ldots\)
g^{-1}(F - f(x)) for any x \in X. By Theorem 3.4.17, F - g(x) and F - f(x) are closed subsets of (G, \sigma). Since f and g are \omega-continuous mappings, f^{-1}(F - g(x)) and f^{-1}(F - f(x)) are \omega-closed subsets of (X, \tau). By Theorem 2.2.20, f^{-1}(F - g(x)) \cap g^{-1}(F - f(x)) is also an \omega-closed set in (X, \tau). Therefore f \oplus g is \omega-continuous and hence f \oplus g \in G^X_w. We can also verify that (G^X_w, \oplus) is a group.

Fix an \omega-irresolute map \phi : (G, \sigma) \to (X, \tau). Let f and g be any two members of G^X_w. Consider the composition of maps f \circ \phi \circ g. Clearly, f \circ \phi \circ g is an \omega-continuous map. Therefore, we can define a binary operation \ast_\phi on G^X_w by f \ast_\phi g = f \circ \phi \circ g for any f, g \in G^X_w. This binary operation \ast_\phi on G^X_w is associative and so (G^X_w, \ast_\phi) is a semi-group.

Further, for any f, g, h \in G^X_w, (f \oplus g) \ast_\phi h = (f \oplus g) \circ \phi \circ h = ((f + g) \circ \phi) \circ h = (f \circ \phi + g \circ \phi) \circ h = f \circ \phi \circ h + g \circ \phi \circ h = f \ast_\phi h + g \ast_\phi h. Therefore (G^X_w, \oplus, \ast_\phi) is a sandwich type right near-ring.

**Corollary 3.4.19** Let (X, \tau) be a T_\omega space. Then (C(X, G), \oplus, \ast_\phi) and (G^X_w, \oplus, \ast_\phi) are identical.

**Proof:** Follows from Remark 3.2.10.

**Corollary 3.4.20** C(X,G) \subseteq G^X_w.

**Proof:** Follows from Proposition 3.2.3.

Next we consider I G^X_w = \{f : (X, \tau) \to (G, \sigma) is \omega-irresolute\}. We shall make I G^X_w also a sandwich type right near-ring.

**Theorem 3.4.21[19]** If f : (X, \tau) \to (Y, \sigma) is continuous and open, then f is pre-semi-open.

**Proposition 3.4.22** Let (X, \ast, \tau) be a topological group and A \subseteq X. If b \in X, then b \ast A and A \ast b are \omega-closed if and only if A is \omega-closed.

**Proof:** Suppose A is \omega-closed in (X, \tau). Define f : (X, \tau) \to (X, \tau) by f(x) = b^{-1} \ast x for all x \in X. Then by the corollary 3.4.16, f is a
homeomorphism and hence by Theorem 3.4.21, \( f \) is pre-semi-open. Since every continuous map is \( \omega \)-continuous, \( f \) is bijective, pre-semi-open and \( \omega \)-continuous and so by the Proposition 3.3.10, \( f \) is \( \omega \)- irresolute. Therefore, \( f^{-1}(A) \) is an \( \omega \)-closed subset of \((X, \tau)\). Since \( f^{-1}(A) = b \cdot A \), \( b \cdot A \) is \( \omega \)-closed. Similarly, it can be shown that \( A \cdot b \) is also \( \omega \)-closed.

The converse part is trivial.

**Theorem 3.4.23** If \((X, \tau)\) is a topological space and \((G, +, \sigma)\) is a topological group with \((X, \tau) \neq (G, \sigma)\), then \( IG_\omega^X = \{f: (X, \tau \rightarrow (G, \sigma)\text{ is } \omega\text{-irresolute}\} \) can be made into a sandwich type right near-ring.

**Proof:** Define \( \oplus : IG_\omega^X \times IG_\omega^X \rightarrow IG_\omega^X \) by \( f \oplus g = f + g \) for all \( f, g \in IG_\omega^X \), where \( (f + g)(x) = f(x) + g(x) \) for all \( x \in X \). Then by using Proposition 3.4.22, we can prove that \((IG_\omega^X, \oplus)\) is a group.

Fix a \( \omega \)-irresolute map \( \phi : (G, \sigma) \rightarrow (X, \tau) \). Define \( \ast_\phi : IG_\omega^X \times IG_\omega^X \rightarrow IG_\omega^X \) by \( f \ast_\phi g = f \circ \phi \circ g \) for all \( f, g \in IG_\omega^X \). Then \( \ast_\phi \) is well-defined, closed and associative and so \((IG_\omega^X, \ast_\phi)\) is a semi-group.

Moreover, for any \( f, g \in IG_\omega^X \), \((f \oplus g) \ast_\phi h = (f \ast_\phi h) \oplus (g \ast_\phi h) \). Thus \((IG_\omega^X, \oplus, \ast_\phi)\) is a sandwich type right near-ring.

**Corollary 3.4.24** \( IG_\omega^X \leq G_\omega^X \).

**Proof:** Follows from Proposition 3.3.6.

We finally prove an isomorphism theorem for sandwich type near-rings.

**Theorem 3.4.25** If \((IG_\omega^X, \oplus, \ast_\phi)\) and \((IH_\omega^Y, \oplus, \ast_\psi)\) be any two sandwich near-rings, \( h : X \rightarrow Y \) be a map such that both \( h \) and \( h^{-1} \) are \( \omega \)- irresolute, \( t : G \rightarrow H \) is a group isomorphism and both \( t \) and \( t^{-1} \) are \( \omega \)- irresolute such that \( h \circ \phi = \psi \circ t \), then \( IG_\omega^X \cong IH_\omega^Y \) as near-rings.

**Proof:** Define \( \alpha : IG_\omega^X \rightarrow IH_\omega^Y \) by \( \alpha(f) = t \circ f \circ h^{-1} \) for all \( f \in IG_\omega^X \).
Clearly $\alpha$ is well defined and bijective. Further for any $f, g \in IG_x \times$,
$$
\alpha(f \oplus g) = t \cdot (f \oplus g) \cdot h^{-1} = t \cdot ((f \oplus g) \cdot h^{-1})
$$
$$
= t \cdot (f \cdot h^{-1} \oplus g \cdot h^{-1}) = t \cdot f \cdot h^{-1} \oplus t \cdot g \cdot h^{-1}
$$
$$
= \alpha(f) \oplus \alpha(g).
$$

Also,
$$
\alpha(f ** g) = t \cdot (f ** g) \cdot h^{-1} = t \cdot (f \cdot \phi \cdot g) \cdot h^{-1}
$$
$$
= t \cdot f \cdot \phi \cdot g \cdot h^{-1} = t \cdot f \cdot (h^{-1} \cdot g \cdot h^{-1})
$$
$$
= (t \cdot f \cdot h^{-1}) \cdot (h \cdot \phi) \cdot (g \cdot h^{-1}) = (t \cdot f \cdot h^{-1}) \cdot (\psi \cdot t) \cdot (g \cdot h^{-1})
$$
$$
= (t \cdot f \cdot h^{-1}) \cdot \psi \cdot (t \cdot g \cdot h^{-1}) = \alpha(f) \cdot \psi \cdot \alpha(g)
$$
$$
= \alpha(f) \cdot \psi \cdot \alpha(g).
$$

Therefore, $IG_x \times \cong IH_\omega \times$.