CHAPTER II

\(\omega\)-CLOSED SETS IN TOPOLOGICAL SPACES

2.1. Introduction

Generalized closed sets form a strong tool in the characterization of topological spaces satisfying weak separation axioms. The concept of generalization was initiated by Levine [58] in 1963. Since then several topologists ([1], [2], [3], [13], [30], [32], [33], [34], [42], [43], [59], [64], [69], [70], [95], [113] and [121]) have contributed to the development of generalizations of closed sets in topological spaces. In 1970, Levine [59] introduced the concept of generalized closed sets in topological spaces and a class of topological spaces called \(T_{1/2}\) spaces. Dunham [39, 41], Dunham and Levine [40] have further investigated some properties of \(T_{1/2}\) spaces and generalized closed sets. Recently Veera kumar [121] introduced a weak form of closed sets namely \(g^*\)-closed sets between closed sets and generalized closed sets in topological spaces and investigated some of their properties. He also introduced and studied four new spaces using \(g^*\)-closed sets, called \(T^*_{1/2}\) spaces, \(\hat{T}_{1/2}\) spaces, \(\alpha T_c\) spaces and \(T_c\) spaces.

In this chapter by using semi-open sets, we introduce a new class of sets in topological spaces, namely \(\omega\)-closed sets, which is properly placed between the class of closed sets and the class of generalized closed sets. The complement of an \(\omega\)-closed set is called an \(\omega\)-open set and we prove that the class of \(\omega\)-open sets form a topology on \(X\). Moreover as applications, we introduce four new spaces namely \(T_\omega\) spaces, \(g T_\omega\) spaces, \(\alpha T_\omega\) spaces and \(wg T_\omega\) spaces. Using these spaces, we obtain some new characterizations for the spaces \(T_{1/2}, \alpha T_b\) and \(T_{wg}\).
2.2. \( \omega \)-closed sets and their basic properties

In this section, we introduce a new class of sets, called \( \omega \)-closed sets in topological spaces and investigate certain basic properties of these sets.

**Definition 2.2.1** A subset \( A \) of \((X, \tau)\) is called an \( \omega \)-closed set if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\).

We denote the class of \( \omega \)-closed sets in \((X, \tau)\) by \( \mathcal{C} \), i.e., \( \mathcal{C} = \{ A \subseteq X : A \text{ is } \omega \text{-closed in } (X, \tau) \} \).

First we prove that the class of \( \omega \)-closed sets properly lies between the class of closed sets and the class of \( g \)-closed sets.

**Proposition 2.2.2** Every closed set is \( \omega \)-closed but not conversely.

*Proof*: Let \( A \) be any closed set and \( U \) be any semi-open set such that \( A \subseteq U \). Then \( \text{cl}(A) \subseteq U \), since \( \text{cl}(A) = A \) and hence \( A \) is \( \omega \)-closed.

**Example 2.2.3** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{b, c\}, X\} \). Then the set \( \{a, b\} \) is \( \omega \)-closed but not closed in \((X, \tau)\).

**Proposition 2.2.4** Every \( \omega \)-closed set is \( g \)-closed but not conversely.

*Proof*: Let \( A \in \mathcal{C} \) and \( U \) be any open set such that \( A \subseteq U \). Since every open set is semi-open and \( A \) is an \( \omega \)-closed set, we have \( \text{cl}(A) \subseteq U \) and hence \( A \) is \( g \)-closed.

**Example 2.2.5** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, X\} \). Then the set \( \{b\} \) is \( g \)-closed but not \( \omega \)-closed in \((X, \tau)\).

**Proposition 2.2.6** Every \( \omega \)-closed set is \( sg \)-closed and hence \( \beta \)-closed but not conversely.

*Proof*: Let \( A \in \mathcal{C} \) and \( U \) be any semi-open set containing \( A \). Then \( \text{scl}(A) \subseteq \text{cl}(A) \subseteq U \), since \( A \) is \( \omega \)-closed. Therefore \( A \) is \( sg \)-closed.

In [30], it has been proved that every \( sg \)-closed set is semi-preclosed (\( = \beta \)-closed). Hence every \( \omega \)-closed set is \( \beta \)-closed.
The set \( \{c\} \) in Example 2.2.5 is sg-closed and \( \beta \)-closed but not \( \omega \)-closed in \((X, \tau)\).

**Proposition 2.2.7** Every \( \omega \)-closed set is \( g\alpha \)-closed and hence pre-closed but not conversely.

**Proof:** Let \( A \in \mathcal{C} \) and \( U \) be any \( \alpha \)-open set containing \( A \). Since every \( \alpha \)-open set is semi-open and since \( \alpha \text{cl}(A) \subseteq \text{cl}(A) \), we have by hypothesis, \( \alpha \text{cl}(A) \subseteq \text{cl}(A) \subseteq U \) and so \( A \) is \( g\alpha \)-closed.

In [30], it has been proved that every \( g\alpha \)-closed set is pre-closed. Therefore every \( \omega \)-closed set is pre-closed.

In Example 2.2.5, the set \( \{b\} \) is both \( g\alpha \)-closed and pre-closed but not \( \omega \)-closed in \((X, \tau)\).

**Proposition 2.2.8** Every \( \omega \)-closed set is \( g\)-closed, \( gsp \)-closed, \( rg \)-closed, \( gp \)-closed, \( gpr \)-closed, \( wg \)-closed, \( \alpha g \)-closed and hence \( \alpha^{**}g \)-closed but not conversely.

**Proof:** Let \( A \in \mathcal{C} \). Then by Proposition 2.2.4, \( A \) is \( g \)-closed. From the investigation of Maki et al [70], we have \( A \) is \( \alpha g \)-closed and so \( \alpha^{**}g \)-closed. Also from the investigations of Dontchev [30], Gnanambal [49] and Nagaveni [79], we know that every \( g \)-closed set is \( g\)-closed, \( gsp \)-closed, \( rg \)-closed, \( gp \)-closed, \( gpr \)-closed and \( wg \)-closed. Thus again by Proposition 2.2.4, \( A \) is \( g\)-closed, \( gsp \)-closed \( rg \)-closed, \( gp \)-closed, \( gpr \)-closed and \( wg \)-closed.

Let \((X, \tau)\) be the topological space as in Example 2.2.5. Then the set \( A = \{c\} \) is \( \alpha g \)-closed, \( \alpha^{**}g \)-closed, \( g\)-closed, \( gsp \)-closed, \( rg \)-closed, \( gp \)-closed, \( gpr \)-closed and \( wg \)-closed but \( A \notin \mathcal{C} \).

**Remark 2.2.9** The following examples show that \( \omega \)-closed sets are independent of \( g^{*} \)-closed sets, \( \alpha \)-closed sets, semi-closed sets, \( \theta \)-generalized closed sets and \( \delta \)-generalized closed sets.
**Example 2.2.10** Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, c\}, X\}$. Then the set 
$\{a, b\}$ is $g^*$-closed, $\theta$-generalized closed and $\delta$-generalized closed but not 
$\omega$-closed. Also the set $\{c\}$ is $\alpha$-closed and semi-closed but not $\omega$-closed.

**Example 2.2.11** Let $(X, \tau)$ be the topological space of Example 2.2.3. Then 
the set $\{b\}$ is $\omega$-closed but not $g^*$-closed, $\alpha$-closed and not semi-closed.

**Example 2.2.12** Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. 
Then the set $A = \{c\}$ is closed and so $\omega$-closed. But it is neither a 
$\theta$-generalized closed set nor a $\delta$-generalized closed set since, if $U = \{a, c\}$, 
then $X = \text{cl}_\theta(A) = \text{cl}_\delta(A) \nsubseteq U \in \tau$.

**Remark 2.2.13** From the above discussions and known results we have the 
following implications:

\[ \begin{align*}
\theta\text{-closed} & \quad \Rightarrow \quad \theta\text{-g-closed} \quad \Rightarrow \quad \text{rg-closed} \\
\delta\text{-closed} & \quad \Rightarrow \quad \delta\text{-g-closed} \quad \Rightarrow \quad \text{gpr-closed} \\
g^*\text{-closed} & \quad \Rightarrow \quad \text{g-closed} \quad \Rightarrow \quad \text{wg-closed} \quad \Leftarrow \quad \alpha g\text{-closed} \\
\omega\text{-closed} & \\
\alpha\text{-closed} \quad \Rightarrow \quad \text{g}\alpha\text{-closed} \quad \Rightarrow \quad \text{gs-closed} \\
\text{semi-closed} \quad \Rightarrow \quad \text{sg-closed} \quad \Rightarrow \quad \text{gsp-closed} \\
\beta\text{-closed} & \quad \Leftarrow \quad \text{preclosed}
\end{align*} \]

**Proposition 2.2.14** If $A, B \in \mathcal{C}$, then their union $A \cup B \in \mathcal{C}$.

**Proof:** Suppose that $A \cup B \subseteq U$ and $U$ is semi-open, then $A \subseteq U$ and
B \subseteq U. Since A and B are \(\omega\)-closed, \(\text{cl}(A) \subseteq U\) and \(\text{cl}(B) \subseteq U\) and hence \(\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B) \subseteq U\). Thus \(A \cup B \in \mathcal{C}\).

Bhattacharyya and Lahiri [13] introduced the class of semi-generalized closed sets (sg-closed sets) and the problem of determining whether the intersection of two sg-closed sets is again sg-closed or not remained unsolved. But recently Dontchev and Maki [32] solved this problem by proving that arbitrary intersection of sg-closed sets is sg-closed. Following the technique of Dontchev and Maki, we next prove that arbitrary intersection of \(\omega\)-closed sets is again \(\omega\)-closed.

**Lemma 2.2.15 [53]** Let \(x\) be a point of \((X, \tau)\). Then \(\{x\}\) is either nowhere dense or preopen.

**Remark 2.2.16[32]** In the notion of Lemma 2.2.15, we may consider the following decomposition of a given topological space \((X, \tau)\), namely \(X = X_1 \cup X_2\), where \(X_1 = \{x \in X : \{x\}\) is nowhere dense\} and \(X_2 = \{x \in X : \{x\}\) is preopen\}.

**Lemma 2.2.17** A subset \(A\) of \((X, \tau)\) is \(\omega\)-closed if and only if \(\text{cl}(A) \subseteq \text{sker}(A)\).

**Proof:** Suppose that \(A\) is \(\omega\)-closed. Then \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open. Let \(x \in \text{cl}(A)\). If \(x \notin \text{sker}(A)\), then there is a semi-open set \(U\) containing \(A\) such that \(x \notin U\). Since \(U\) is a semi-open set containing \(A\), we have \(x \notin \text{cl}(A)\), a contradiction.

Conversely, let \(\text{cl}(A) \subseteq \text{sker}(A)\). If \(U\) is any semi-open set containing \(A\), then \(\text{cl}(A) \subseteq \text{sker}(A) \subseteq U\). Therefore \(A\) is \(\omega\)-closed.

**Proposition 2.2.18** For any subset \(A\) of \((X, \tau)\), \(X_2 \cap \text{cl}(A) \subseteq \text{sker}(A)\).

**Proof:** Let \(x \in X_2 \cap \text{cl}(A)\) and suppose that \(x \notin \text{sker}(A)\). Then there is a semi-open set \(U\) containing \(A\) such that \(x \notin U\). If \(F = X - U\), then \(F\) is semi-closed and so \(\text{scl}(\{x\}) = \{x\} \cup \text{int}(\text{cl}(\{x\})) \subseteq F\). Since \(\text{cl}(\{x\}) \subseteq \text{cl}(A)\), we have \(\text{int}(\text{cl}(\{x\})) \subseteq A \cup \text{int}(\text{cl}(A))\). Again since \(x \in X_2\), we
have $x \notin X_1$ and so $\text{int}(\text{cl} \{x\}) \neq \emptyset$. Therefore there has to be some point $y \in A \cap \text{int}(\text{cl} \{x\})$ and hence $y \in F \cap A$, a contradiction.

**Theorem 2.2.19** A subset $A$ of $(X, \tau)$ is $\omega$-closed if and only if $X_1 \cap \text{cl}(A) \subseteq A$.

**Proof**: Suppose that $A$ is $\omega$-closed. Let $x \in X_1 \cap \text{cl}(A)$. Then $x \in X_1$ and $x \in \text{cl}(A)$. Since $x \in X_1$, $\text{int}(\text{cl} \{x\}) = \emptyset$. Therefore $\{x\}$ is semi-closed, since $\text{int}(\text{cl} \{x\}) \subseteq \{x\}$. If $x \notin A$ and if $U = X - \{x\}$, then $U$ is a semi-open set containing $A$ and so $\text{cl}(A) \subseteq U$, a contradiction.

Conversely, suppose that $X_1 \cap \text{cl}(A) \subseteq A$. Then $X_1 \cap \text{cl}(A) \subseteq \text{sker}(A)$, since $A \subseteq \text{sker}(A)$. Now, $\text{cl}(A) = X \cap \text{cl}(A) = (X_1 \cup X_2) \cap \text{cl}(A) = (X_1 \cap \text{cl}(A)) \cup (X_2 \cap \text{cl}(A)) \subseteq \text{sker}(A)$, since $X_1 \cap \text{cl}(A) \subseteq \text{sker}(A)$ and by Proposition 2.2.18. Thus $A$ is $\omega$-closed by Lemma 2.2.17.

**Theorem 2.2.20** An arbitrary intersection of $\omega$-closed sets is $\omega$-closed.

**Proof**: Let $F = \{A_i : i \in \Lambda\}$ be a family of $\omega$-closed sets and let $A = \cap_{i \in \Lambda} A_i$. Since $A \subseteq A_i$ for each $i$, $X_1 \cap \text{cl}(A) \subseteq X_1 \cap \text{cl}(A_i)$ for each $i$.

Using Theorem 2.2.19 for each $\omega$-closed set $A_i$, we have $X_1 \cap \text{cl}(A_i) \subseteq A_i$ for each $i$, and so $X_1 \cap \text{cl}(A_i) \subseteq A$ for each $i$. Thus $X_1 \cap \text{cl}(A) \subseteq X_1 \cap \text{cl}(A_i) \subseteq A$ for each $i \in \Lambda$. i.e., $X_1 \cap \text{cl}(A) \subseteq A$ and so $A$ is $\omega$-closed by Theorem 2.2.19.

**Corollary 2.2.21** If $A$ is an $\omega$-closed set and $F$ is a closed set, then $A \cap F$ is an $\omega$-closed set.

**Proof**: Since $F$ is closed, it is $\omega$-closed. Therefore by Theorem 2.2.20, $A \cap F$ is also an $\omega$-closed set.

**Proposition 2.2.22** If a subset $A$ of $(X, \tau)$ is $\omega$-closed, then $\text{cl}(A) - A$ contains no nonempty closed set.

**Proof**: Suppose that $A$ is $\omega$-closed in $(X, \tau)$ and $F$ be a closed subset of $\text{cl}(A) - A$. Then $A \subseteq F^c$. Since $F^c$ is semi-open and $A$ is $\omega$-closed, $\text{cl}(A)$
Consequently, \( F \subseteq (\text{cl}(A))^c \). We have \( F \subseteq \text{cl}(A) \). Thus \( F \subseteq \text{cl}(A) \cap (\text{cl}(A))^c = \emptyset \) and hence \( F \) is empty.

**Remark 2.2.23** The converse of the above theorem is not true in general. Consider the topological space \((X, \tau)\) of Example 2.2.5. Let \( A = \{b\} \), then \( \text{cl}(A) - A = \{c\} \) does not contain non empty closed sets. But \( A \) is not \( \omega \)-closed in \((X, \tau)\).

**Corollary 2.2.24** In a \(T_1\) space, \(\omega\)-closed sets are closed.

**Proof:** Let \( A \) be a \(\omega\)-closed set in a \(T_1\) space \((X, \tau)\). If \( x \in \text{cl}(A) - A \), then \( \{x\} \subseteq \text{cl}(A) - A \) and since \( X \) is \(T_1\), \( \{x\} \) is a closed set in \((X, \tau)\). By Proposition 2.2.22, there exists no element in \(\text{cl}(A) - A\) and so \(\text{cl}(A) - A = \emptyset\). Therefore \(\text{cl}(A) = A\). i.e., \( A \) is closed.

**Theorem 2.2.25** \( i) \) \( A \) is semi-open if and only if \( \text{sint}(A) = A \) and \( ii) \) \( A \) is semi-closed if and only if \( \text{scl}(A) = A \), where \( \text{sint}(A) \) is the semi-interior of \( A \) and \( \text{scl}(A) \) is the semi-closure of \( A \).

**Theorem 2.2.26** \( [18]\) If \( A \subseteq X \), then \( \text{int}(A) \subseteq \text{sint}(A) \subseteq A \) and \( A \subseteq \text{scl}(A) \subseteq \text{cl}(A) \).

**Theorem 2.2.27** If \( C \) is closed and \( T \) is semi-closed, then \( C \cap T \) is semi-closed.

**Proof:** Since \( C \) is closed, it is semi-closed and therefore \( C \cap T \) is semi-closed \([18]\).

**Theorem 2.2.28** A subset \( A \) of \((X, \tau)\) is \(\omega\)-closed if and only if \(\text{cl}(A) - A\) does not contain any nonempty semi-closed set.

**Proof:** Suppose that \( A \) is \(\omega\)-closed. Let \( U \) be a semi closed subset of \(\text{cl}(A) - A\). Then \( A \subseteq U^c \). Since \( A \) is \(\omega\)-closed, we have \(\text{cl}(A) \subseteq U^c \). Consequently, \( U \subseteq (\text{cl}(A))^c \). Hence \( U \subseteq \text{cl}(A) \cap (\text{cl}(A))^c = \emptyset \). Therefore \( U \) is empty.

Conversely, suppose that \(\text{cl}(A) - A\) contains no nonempty semi-closed set. Let \( A \subseteq U \) and that \( U \) be semi-open. If \(\text{cl}(A) \nsubseteq U \), then \(\text{cl}(A) \cap \(...
U^c \neq \emptyset. Since \text{cl}(A) is a closed set and U^c is a semi-closed set of (X, \tau), by Theorem 2.2.27, \text{cl}(A) \cap U^c is a semi-closed set of (X, \tau). Therefore \emptyset \neq \text{cl}(A) \cap U^c \subseteq \text{cl}(A) - A and so \text{cl}(A) - A contains a non-empty semi-closed set, which is a contradiction to the hypothesis. Thus A is an \omega-closed set.

**Corollary 2.2.29** An \omega-closed set A is semi-closed if and only if \text{scl}(A) - A is semi-closed.

**Proof:** Let A be any \omega-closed set. If A is semi closed, then \text{scl}(A) - A = \emptyset, by Theorem 2.2.25. Therefore \text{scl}(A) - A is semi-closed.

Conversely, suppose that \text{scl}(A) - A is semi-closed. By Theorem 2.2.26, \text{cl}(A) - A contains the semi-closed set \text{scl}(A) - A. Since A is \omega-closed, by Theorem 2.2.28, \text{scl}(A) - A = \emptyset. Hence \text{scl}(A) = A. Therefore, A is semi-closed, by Theorem 2.2.25.

**Theorem 2.2.30** In a topological space (X, \tau), the following are equivalent:

i) A is \omega-closed.

ii) For each x \in \text{cl}(A), \text{scl}(x) \cap A \neq \emptyset.

iii) \text{cl}(A) - A contains no non-empty semi-closed sets.

**Proof:** i) \Rightarrow ii): Suppose x \in \text{cl}(A) and \text{scl}(x) \cap A = \emptyset. Then A \subseteq (\text{scl}(x))^c and (\text{scl}(x))^c is semi-open. By assumption, \text{cl}(A) \subseteq (\text{scl}(x))^c, which is a contradiction to x \in \text{cl}(A).

ii) \Rightarrow iii): Let F \subseteq \text{cl}(A) - A, where F is semi-closed. If there is an x \in F, then x \in \text{cl}(A) and so by assumption, \emptyset \neq \text{scl}(x) \cap A \subseteq F \cap A \subseteq (\text{cl}(A) - A) \cap A = \emptyset, a contradiction. Therefore F = \emptyset.

iii) \Rightarrow i): Follows from Theorem 2.2.28.

**Corollary 2.2.31** A is \omega-closed if and only if A = F - N, where F is closed and N contains no nonempty semi-closed set.

**Proof:** Suppose that A is \omega-closed. Let F = \text{cl}(A) and N = \text{cl}(A) - A. Then A = F - N, by Theorem 2.2.30.
Conversely, assume $A = F - N$. Let $A \subseteq U$ where $U$ is any semi-open set. Then $F \cap U^c$ is semi-closed by Theorem 2.2.27 and it is a subset of $N$. By assumption $F \cap U^c = \emptyset$, i.e., $F \subseteq U$. Thus $\text{cl}(A) \subseteq F \subseteq U$ and so $A$ is $\omega$-closed.

**Theorem 2.2.32** If $A$ is open and $S$ is semi-open, then $A \cap S$ is semi-open.

**Theorem 2.2.33** Let $A \subseteq Y \subseteq X$, where $X$ is a topological space and $Y$ is a subspace. Let $A \in \text{SO}(X)$. Then $A \in \text{SO}(Y)$.

**Theorem 2.2.34** Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a collection of semi-open sets in a topological space $(X, \tau)$. Then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is semi-open.

**Proposition 2.2.35** Suppose that $B \subseteq A \subseteq X$, $B$ is an $\omega$-closed set relative to $A$ and that $A$ is open and $\omega$-closed in $(X, \tau)$. Then $B$ is $\omega$-closed in $(X, \tau)$.

**Proof**: Let $B \subseteq U$, where $U$ is any semi-open set in $(X, \tau)$. Then $B \subseteq A \cap U$. Since $A$ is open and $U$ is semi-open, by Theorem 2.2.32, $A \cap U$ is semi-open in $(X, \tau)$. Since $A \cap U \subseteq A \subseteq X$ and $A \cap U$ is semi-open in $(X, \tau)$ by Theorem 2.2.33, $A \cap U$ is semi-open in $A$. So $A \cap U$ is a semi-open set in $A$ such that $B \subseteq A \cap U$. By hypothesis, $B$ is an $\omega$-closed set relative to $A$. Thus $\text{cl}_A(B) \subseteq A \cap U$. Since $\text{cl}_A(B) = A \cap \text{cl}(B)$, we have $A \cap \text{cl}(B) \subseteq A \cap U$, from which we obtain $A \subseteq U \cup (\text{cl}(B))^c$ and $U \cup (\text{cl}(B))^c$ is a semi-open set in $(X, \tau)$, by Theorem 2.2.34. By hypothesis, $A$ is $\omega$-closed in $(X, \tau)$ and therefore $\text{cl}(A) \subseteq U \cup (\text{cl}(B))^c$. Since $\text{cl}(B) \subseteq \text{cl}(A)$, $\text{cl}(B) \subseteq U \cup (\text{cl}(B))^c$ and hence $\text{cl}(B) \subseteq U$. Therefore $B$ is an $\omega$-closed set relative to $(X, \tau)$.

**Proposition 2.2.36** If $A$ is an $\omega$-closed set of $(X, \tau)$ such that $A \subseteq B \subseteq \text{cl}(A)$, then $B$ is also an $\omega$-closed set of $(X, \tau)$.

**Proof**: Let $U$ be a semi-open set of $(X, \tau)$ such that $B \subseteq U$. Then $A \subseteq U$. Since $A$ is $\omega$-closed, we have $\text{cl}(A) \subseteq U$. Now $\text{cl}(B) \subseteq \text{cl}(\text{cl}(A)) = \text{cl}(A) \subseteq U$. 

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Therefore, B is also an ω-closed set of (X, τ).

**Remark 2.2.37** The converse of Proposition 2.2.36 need not be true in general. Consider the topological space (X, τ) of Example 2.2.3. Let A = \{a\} and B = \{a, b\}. Then A and B are ω-closed sets in (X, τ) but A ⊆ B ⊈ cl(A).

**Lemma 2.2.38** If \( A \in \text{SO}(X_\circ) \), then \( A = B \cap X_\circ \) for some \( B \in \text{SO}(X) \), where \( X \) is a topological space and \( X_\circ \) is a subspace of \( X \).

**Proposition 2.2.39** Let \( A \subseteq Y \subseteq X \) and suppose that A is ω-closed in (X, τ).

Then A is ω-closed relative to Y.

**Proof:** Let \( A \subseteq S \), where S is any semi-open set in Y. Then by Lemma 2.2.38, \( S = U \cap Y \) for some semi-open set \( U \) in (X, τ). Thus \( A \subseteq U \cap Y \) and so \( A \subseteq U \). Since A is ω-closed in (X, τ), \( \text{cl}(A) \subseteq U \) and therefore \( Y \cap \text{cl}(A) \subseteq Y \cap U \). i.e., \( \text{cl}_Y(A) \subseteq S \), since \( \text{cl}_Y(A) = Y \cap \text{cl}(A) \). Hence A is ω-closed relative to Y.

The converse of Proposition 2.2.39 need not be true as seen from the following example.

**Example 2.2.40** Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{a, b\}, X\} \) and \( Y = \{a, b, c\} \). Then \( \tau_Y = \{\emptyset, \{a\}, \{a, b\}, Y\} \). Take \( A = \{b, c\} \). Then \( A \subseteq Y \subseteq X \) and A is ω-closed relative to Y. But A is not ω-closed in (X, τ).

**Theorem 2.2.41** In a topological space (X, τ), \( \text{SO}(X,\tau) = \{F \subseteq X : F^c \in \tau\} \) if and only if every subset of (X, τ) is an ω-closed set.

**Proof:** Suppose that \( \text{SO}(X) = \{F \subseteq X : F^c \in \tau\} \). Let A be a subset of (X, τ) such that \( A \subseteq U \), where \( U \in \text{SO}(X,\tau) \). Then \( \text{cl}(U) = U \). Also \( \text{cl}(A) \subseteq \text{cl}(U) = U \). Hence A is ω-closed.

Conversely, suppose that every subset of (X, τ) is ω-closed. Let \( U \in \text{SO}(X,\tau) \). Since \( U \subseteq U \) and U is ω-closed, we have \( \text{cl}(U) \subseteq U \). Thus \( \text{cl}(U) = U \) and \( U \in \{F \subseteq X : F^c \in \tau\} \). Therefore \( \text{SO}(X,\tau) \subseteq \{F \subseteq X : F^c \in \tau\} \).
If \( F \in \{ F \subseteq X : F^c \in \tau \} \), then \( F^c \) is semi-open. Therefore \( F^c \in \text{SO}(X, \tau) \subseteq \{ F \subseteq X : F^c \in \tau \} \). Hence \( F \) is open in \((X, \tau)\) and so \( F \) is semi-open in \((X, \tau)\). i.e., \( F \in \text{SO}(X, \tau) \). Thus \( \text{SO}(X, \tau) = \{ F \subseteq X : F^c \in \tau \} \).

**Proposition 2.2.42** If \( A \) is semi-open and \( \omega \)-closed, then \( A \) is closed.

**Proof:** Since \( A \subseteq A \) and \( A \) is semi-open and \( \omega \)-closed, we have \( \text{cl}(A) \subseteq A \). Therefore, \( \text{cl}(A) = A \) and \( A \) is closed.

**Proposition 2.2.43** For each \( x \in X \), either \( \{ x \} \) is semi-closed or \( \{ x \}^c \) is \( \omega \)-closed in \((X, \tau)\).

**Proof:** Suppose that \( \{ x \} \) is not semi-closed in \((X, \tau)\). Then \( \{ x \}^c \) is not semi-open and the only semi-open set containing \( \{ x \}^c \) is the space \( X \) itself. Therefore, \( \text{cl}(\{ x \}^c) \subseteq X \) and so \( \{ x \}^c \) is \( \omega \)-closed.

**Proposition 2.2.44** Let \((X, \tau)\) be a compact topological space and suppose that \( A \) is an \( \omega \)-closed subset of \((X, \tau)\). Then \( A \) is compact.

**Proof:** Suppose that \( C \) be an open cover of \( A \). Since \( \bigcup_{G \in C} G \) is semi-open by Theorem 2.2.34 and \( A \) is \( \omega \)-closed, we have \( \text{cl}(A) \subseteq \bigcup_{G \in C} G \). Also \( \text{cl}(A) \) is compact in \((X, \tau)\). Therefore \( A \subseteq \text{cl}(A) \subseteq \bigcup_{i=1}^n G_i \), where \( G_1, G_2, \ldots, G_n \in C \). Hence \( A \) is compact.

**Proposition 2.2.45** Let \((X, \tau)\) be a normal space and suppose that \( Y \) is an \( \omega \)-closed subset of \((X, \tau)\). Then the subspace \( Y \) is normal.

**Proof:** Suppose that \( F_1 \) and \( F_2 \) are closed sets in \((X, \tau)\) such that \((Y \cap F_1) \cap (Y \cap F_2) = \emptyset\). Then \( Y \subseteq (F_1 \cap F_2)^c \) and \((F_1 \cap F_2)^c \) is semi-open. But \( Y \) is \( \omega \)-closed in \((X, \tau)\). Therefore, \( \text{cl}(Y) \subseteq (F_1 \cap F_2)^c \) and hence \( \text{cl}(Y) \cap F_1 \cap (\text{cl}(Y) \cap F_2) = \emptyset \). Since \((X, \tau)\) is normal, there exist disjoint open sets \( G_1 \) and \( G_2 \) such that \( \text{cl}(Y) \cap F_1 \subseteq G_1 \) and \( \text{cl}(Y) \cap F_2 \subseteq G_2 \). Thus \( Y \cap G_1 \) and \( Y \cap G_2 \) are two disjoint open sets of \( Y \) such that \( Y \cap F_1 \subseteq Y \cap G_1 \) and \( Y \cap F_2 \subseteq Y \cap G_2 \). Therefore, \( Y \) is normal.

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Proposition 2.2.46 If \((X, \tau)\) is semi-normal and \(F \cap A = \emptyset\), where \(F\) is semi-closed and \(A\) is \(\omega\)-closed then there exists disjoint semi-open sets \(U_1\) and \(U_2\) such that \(F \subseteq U_1\) and \(A \subseteq U_2\).

Proof: Since \(F\) is semi-closed and \(F \cap A = \emptyset\) we have \(A \subseteq F^c\) and so \(\text{cl}(A) \subseteq F^c\). Thus \(\text{cl}(A) \cap F = \emptyset\). Since \(\text{cl}(A)\) and \(F\) are semi-closed and \((X, \tau)\) is semi-normal there exists semi-open sets \(U_1\) and \(U_2\) such that \(\text{cl}(A) \subseteq U_1\) and \(F \subseteq U_2\), i.e., \(A \subseteq U_1\) and \(F \subseteq U_2\).

Remark 2.2.47 Disjoint \(\omega\)-closed sets in a semi-normal space cannot be separated by semi-open sets as seen from the following example.

Example 2.2.48 Let \((X, \tau)\) be the topological space of Example 2.2.3. Then \((X, \tau)\) is a semi-normal space, but \(\{a, b\}\) and \(\{c\}\) are disjoint \(\omega\)-closed sets which cannot be separated by disjoint semi-open sets in \((X, \tau)\).

Proposition 2.2.49 If \((X, \tau)\) is \(s\)-normal and \(F \cap A = \emptyset\), where \(F\) is closed and \(A\) is \(\omega\)-closed, then there exists disjoint semi-open sets \(U_1\) and \(U_2\) such that \(F \subseteq U_1\) and \(A \subseteq U_2\).

Proof: Similar to Proposition 2.2.46.

2.3. \(\omega\)-closure

In this section, we define \(\omega\)-closure of a set and we prove that \(\omega\)-closure is a Kuratowski closure operator on \(X\).

Definition 2.3.1 For every set \(E \subseteq X\), we define the \(\omega\)-closure of \(E\) to be the intersection of all \(\omega\)-closed sets containing \(E\).

In symbols, \(\omega\text{-cl}(E) = \cap \{A : E \subseteq A \in \mathcal{C}\}\).

Lemma 2.3.2 For any \(E \subseteq X\), \(E \subseteq \omega\text{-cl}(E) \subseteq \text{cl}(E)\).
Proof: Follows from Proposition 2.2.2.

Remark 2.3.3 Both containment relations in Lemma 2.3.2 may be proper as seen from the following example.

Example 2.3.4 Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a, b\}, X\}$. Let $E = \{a\}$. Then $\omega-\text{cl}(E) = \{a, c\}$ and so $E \subseteq \omega-\text{cl}(E) \subseteq \text{cl}(E)$.

Lemma 2.3.5 For any $E \subseteq X$, $C^\ast(E) \subseteq \omega-\text{cl}(E)$, where $C^\ast(E)$ is given by $C^\ast(E) = \cap \{A : E \subseteq A, A \in GC(X, \tau)\}$.

Proof: Follows from Proposition 2.2.4.

Theorem 2.3.6 $\omega$-closure is a Kuratowski closure operator on $X$.

Proof: i). $\omega-\text{cl}(\phi) = \phi$ and ii). $E \subseteq \omega-\text{cl}(E)$, by Lemma 2.3.2.

iii). Let $E_1 \cup E_2 \subseteq A \in \mathcal{C}$, then $E_i \subseteq A$ and by Definition 2.3.1, $\omega-\text{cl}(E_i) \subseteq A$ for $i = 1, 2$. Therefore, $\omega-\text{cl}(E_1) \cup \omega-\text{cl}(E_2) \subseteq \cap \{A : E_1 \cup E_2 \subseteq A \in \mathcal{C}\} = \omega-\text{cl}(E_1 \cup E_2)$. To prove the reverse inclusion, let $x \in \omega-\text{cl}(E_1 \cup E_2)$ and suppose that $x \notin \omega-\text{cl}(E_1) \cup \omega-\text{cl}(E_2)$. Then there exists $\omega$-closed sets $A_1$ and $A_2$ with $E_1 \subseteq A_1$, $E_2 \subseteq A_2$ and $x \notin A_1 \cup A_2$. We have $E_1 \cup E_2 \subseteq A_1 \cup A_2$ and $A_1 \cup A_2$ is an $\omega$-closed set by Proposition 2.2.14 such that $x \notin A_1 \cup A_2$. Thus $x \notin \omega-\text{cl}(E_1 \cup E_2)$, which is a contradiction to $x \in \omega-\text{cl}(E_1 \cup E_2)$. Hence $\omega-\text{cl}(E_1 \cup E_2) = \omega-\text{cl}(E_1) \cup \omega-\text{cl}(E_2)$.

iv). Let $E \subseteq A \in \mathcal{C}$. Then by Definition 2.3.1, $\omega-\text{cl}(E) \subseteq A$ and $\omega-\text{cl}(\omega-\text{cl}(E)) \subseteq A$. Since $\omega-\text{cl}(\omega-\text{cl}(E)) \subseteq A$, we have $\omega-\text{cl}(\omega-\text{cl}(E)) \subseteq \cap \{A : E \subseteq A \in \mathcal{C}\} = \omega-\text{cl}(E)$. By Lemma 2.3.2, $\omega-\text{cl}(E) \subseteq \omega-\text{cl}(\omega-\text{cl}(E))$ and therefore, $\omega-\text{cl}(\omega-\text{cl}(E)) = \omega-\text{cl}(E)$.

Hence, $\omega$-closure is a Kuratowski closure operator on $X$.

Definition 2.3.7 Let $\tau_\omega$ be the topology on $X$ generated by $\omega$-closure in the usual manner. i.e., $\tau_\omega = \{U : \omega-\text{cl}(U^\circ) = U^\circ\}$.

Proposition 2.3.8 For any topology $\tau$, $\tau \subseteq \tau_\omega \subseteq \tau^\ast$, where $\tau^\ast = \{U / C^\ast(U^\circ) = U^\circ\}$.
Proof: Follows from Propositions 2.2.2, 2.2.4, Lemma 2.3.5.
The following two propositions are easy consequences from definitions.

**Proposition 2.3.9** For any \( E \subseteq X \),

i) \( \omega\text{-cl} (E) \) is the smallest \( \omega \)-closed set containing \( E \),

ii) \( E \) is \( \omega \)-closed if and only if \( \omega\text{-cl}(E) = E \).

**Proposition 2.3.10** For any two subsets \( E_1 \) and \( E_2 \) of \((X, \tau)\),

i) If \( E_1 \subseteq E_2 \), then \( \omega\text{-cl} (E_1) \subseteq \omega\text{-cl}(E_2) \),

ii) \( \omega\text{-cl}(E_1 \cap E_2) \subseteq \omega\text{-cl}(E_1) \cap \omega\text{-cl}(E_2) \).

---

### 2.4. \( \omega \)-open sets

In this section, we define \( \omega \)-open sets and \( \omega \)-interior in topological spaces and prove that \( \omega \)-open sets form a topology on \( X \).

**Definition 2.4.1** A subset \( A \) in \((X, \tau)\) is called \( \omega \)-open in \( X \) if \( A^c \) is \( \omega \)-closed in \((X, \tau)\).

We denote the family of all \( \omega \)-open sets in \((X, \tau)\) by \( \tau^\omega \).

The following three propositions are the analogue of Propositions 2.2.2, 2.2.4, 2.2.6 to 2.2.8, 2.2.14 and Theorem 2.2.20 for \( \omega \)-open sets.

**Proposition 2.4.2** For any topological space \((X, \tau)\), we have the following:

i) Every open set is \( \omega \)-open.

ii) Every \( \omega \)-open set is \( g \)-open.

iii) Every \( \omega \)-open set is \( sg \)-open and hence \( \beta \)-open.

iv) Every \( \omega \)-open set is \( ga \)-open and hence preopen.

v) Every \( \omega \)-open set is \( gs \)-open, \( gsp \)-open, \( rg \)-open, \( gp \)-open, \( gpr \)-open, \( ag \)-open, and hence \( \alpha \)-g-open.
**Proposition 2.4.3** If $A, B \in \tau^o$, then $A \cap B \in \tau^o$

**Proposition 2.4.4** An arbitrary union of $\omega$-open sets is $\omega$-open

**Remark 2.4.5** Since $\phi, X \in \tau^o$ always and by Propositions 2.4.3 and 2.4.4, we conclude that $\tau^o$ is a topology on $X$.

**Theorem 2.4.6** A set $A$ is $\omega$-open iff $F \subseteq \text{int}(A)$ whenever $F$ is semi-closed and $F \subseteq A$.

**Proof:** Suppose that $F \subseteq \text{int}(A)$, where $F$ is semi-closed and $F \subseteq A$. Let $A^c \subseteq U$, where $U$ is semi-open. Then $U^c \subseteq A$ and $U^c$ is semi-closed. Therefore $U^c \subseteq \text{int}(A)$. Since $U^c \subseteq \text{int}(A)$, we have $(\text{int}(A))^c \subseteq U$. i.e., $\text{cl}(A^c) \subseteq U$, since $\text{cl}(A^c) = (\text{int}(A))^c$. Thus $A^c$ is $\omega$-closed. i.e., $A$ is $\omega$-open.

Conversely, suppose that $A$ is $\omega$-open, $F \subseteq A$ and $F$ is semi-closed. Then $F^c$ is semi-open and $A^c \subseteq F^c$. Therefore, $\text{cl}(A^c) \subseteq F^c$ and so $F \subseteq \text{int}(A)$, since $\text{cl}(A^c) = (\text{int}(A))^c$.

**Theorem 2.4.7** A set $A$ is $\omega$-open in $(X, \tau)$ iff $U = X$ whenever $U$ is semi-open and $\text{int}(A) \cup A^c \subseteq U$.

**Proof:** Let $A$ be $\omega$-open, $U$ be semi-open and $\text{int}(A) \cup A^c \subseteq U$. This gives $U^c \subseteq (\text{int}(A))^c \cap (A^c)^c = (\text{int}(A))^c - A^c = \text{cl}(A^c) - A^c$. Since $A^c$ is $\omega$-closed and $U^c$ is semi-closed, by Theorem 2.2.28, it follows that $U^c = \phi$. i.e., $X = U$.

Conversely, suppose that $F$ is semi-closed and $F \subseteq A$. Then $\text{int}(A) \cup A^c \subseteq \text{int}(A) \cup F^c$. Since both $\text{int}(A)$ and $F^c$ are semi-open, their union $\text{int}(A) \cup F^c$ is also semi-open by Theorem 2.2.34 and hence by hypothesis, $\text{int}(A) \cup F^c = X$, which implies $F \subseteq \text{int}(A)$. Therefore, $A$ is $\omega$-open by Theorem 2.4.6.

**Proposition 2.4.8** If $A \subseteq B \subseteq X$ where $A$ is $\omega$-open relative to $B$ and $B$ is open in $(X, \tau)$, then $A$ is $\omega$-open in $(X, \tau)$.

**Proof:** Let $F$ be a semi-closed set in $(X, \tau)$ such that $F \subseteq A$. Since $F$ is semi-closed in $(X, \tau)$, there exists a closed set $K$ such that $\text{int}(K) \subseteq F \subseteq K$. 

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Therefore, \( \text{int}(K) \cap B \subseteq F \subseteq K \cap B \). We have, \( \tau_B - \text{int}(K \cap B) \subseteq \text{int}(K) \cap B \) and \( K \cap B \) is \( \tau_B \)-closed and so \( \tau_B - \text{int}(K \cap B) \subseteq \text{int}(K) \cap B \subseteq F \subseteq K \cap B \).

i.e., \( \tau_B - \text{int}(K \cap B) \subseteq F \subseteq K \cap B \), which implies \( F \) is semi-closed in \( B \). Since \( F \) is semi-closed in \( B \) such that \( F \subseteq A \) and \( A \) is \( \omega \)-open relative to \( B \), we have \( F \subseteq \tau_B - \text{int}(A) \), by Theorem 2.4.6. But \( \tau_B - \text{int}(A) \) is an open set relative to \( B \) and so \( F \subseteq U \cap B \subseteq A \) for some open set \( U \) in \( (X, \tau) \). By hypothesis \( B \) is open in \( (X, \tau) \) and therefore \( F \subseteq \text{int}(B) \subseteq B \). Thus \( F \subseteq \text{int}(B) \cap U \subseteq B \cap U \subseteq A \) and \( F \subseteq \text{int}(A) \), which implies that \( A \) is \( \omega \)-open in \( (X, \tau) \).

**Proposition 2.4.9** If \( \text{int}(A) \subseteq B \subseteq A \) and if \( A \) is \( \omega \)-open, then \( B \) is \( \omega \)-open.

**Proof**: Suppose that \( \text{int}(A) \subseteq B \subseteq A \) and \( A \) is \( \omega \)-open. Then \( A^c \subseteq B^c \subseteq \text{cl}(A^c) \) and since \( A^c \) is \( \omega \)-closed, we have by Proposition 2.2.36, \( B^c \) is \( \omega \)-closed. i.e., \( B \) is \( \omega \)-open.

**Proposition 2.4.10** Let \( A \subseteq Y \subseteq X \) and suppose that \( Y \) is closed in \( (X, \tau) \) and \( A \) is \( \omega \)-open in \( (X, \tau) \). Then \( A \) is \( \omega \)-open relative to \( Y \).

**Proof**: Let \( A^c \subseteq U \), where \( U \) is any semi-open set in \( Y \). Then by Lemma 2.2.38, \( U = V \cap Y \) for some semi-open set \( V \) in \( (X, \tau) \) and so \( A^c \subseteq V \) and \( A^c \subseteq Y \). By hypothesis, \( A^c \) is \( \omega \)-closed and \( Y \) is closed in \( (X, \tau) \) and therefore \( \text{cl}(A^c) \subseteq V \) and \( \text{cl}(A^c) \subseteq Y \). i.e., \( \text{cl}(A^c) \subseteq V \cap Y = U \). Thus \( Y \cap \text{cl}(A^c) \subseteq Y \cap U = U \), which implies that \( A^c \) is \( \omega \)-closed in \( Y \). i.e., \( A \) is \( \omega \)-open in \( Y \).

**Remark 2.4.11** Any intersection of semi-closed sets is semi-closed.

**Theorem 2.4.12** A set \( A \) is \( \omega \)-closed iff \( \text{cl}(A) - A \) is \( \omega \)-open.

**Proof**: Suppose that \( A \) is \( \omega \)-closed. Let \( F \subseteq \text{cl}(A) - A \), where \( F \) is semi-closed. By Theorem 2.2.28, \( F = \emptyset \). Therefore \( F \subseteq \text{int}(\text{cl}(A) - A) \) and by Theorem 2.4.6, \( \text{cl}(A) - A \) is \( \omega \)-open.

Conversely, let \( A \subseteq U \) where \( U \) is a semi-open set. Then \( \text{cl}(A) \cap U^c \subseteq \text{cl}(A) \cap A^c = \text{cl}(A) - A \). Since \( \text{cl}(A) \cap U^c \) is semi-closed by Remark 2.4.11,
and $\text{cl}(A) - A$ is $\omega$-open, we have by Theorem 2.4.6, $\text{cl}(A) \cap U^c \subseteq \text{int}(\text{cl}(A) - A) = \emptyset$. Hence $\text{cl}(A) \subseteq U$ and so $A$ is $\omega$-closed.

**Theorem 2.4.13** For a subset $A \subseteq X$ the following are equivalent:

i) $A$ is $\omega$-closed,

ii) $\text{cl}(A) - A$ contains no non-empty semi-closed set,

iii) $\text{cl}(A) - A$ is $\omega$-open.

**Proof:** i) $\iff$ ii) by Theorem 2.2.28 and i) $\iff$ iii) by Theorem 2.4.12.

**Lemma 2.4.14** For an $x \in X$, $x \in \omega\text{-cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $\omega$-open set $U$ containing $x$.

**Proof:** Let $x \in \omega\text{-cl}(A)$ for any $x \in X$. Suppose there exists an $\omega$-open set $U$ containing $x$ such that $U \cap A = \emptyset$. Then $A \subseteq U^c$. Since $U^c$ is an $\omega$-closed set containing $A$, we have $\omega\text{-cl}(A) \subseteq U^c$, which implies that $x \notin \omega\text{-cl}(A)$, a contradiction.

Conversely, suppose that $x \notin \omega\text{-cl}(A)$. Then by Definition 2.3.1, there exists an $\omega$-closed set $F$ containing $A$ such that $x \notin F$. Then $x \in F^c$ and $F^c$ is $\omega$-open. Also $F^c \cap A = \emptyset$, which is a contradiction to the hypothesis.

Therefore $x \in \omega\text{-cl}(A)$.

**Definition 2.4.15** For any $A \subseteq X$, $\omega\text{-int}(A)$ is defined as the union of all $\omega$-open sets contained in $A$. i.e., $\omega\text{-int}(A) = \cup \{U : U \subseteq A \text{ and } U \in \tau^\omega\}$.

**Lemma 2.4.16** For any $A \subseteq X$, $\text{int}(A) \subseteq \omega\text{-int}(A) \subseteq A$.

**Proof:** Follows from Proposition 2.4.2.

The following two propositions are easy consequences from definitions.

**Proposition 2.4.17** For any $A \subseteq X$,

i) $\omega\text{-int}(A)$ is the largest $\omega$-open set contained in $A$,

ii) $A$ is $\omega$-open if and only if $\omega\text{-int}(A) = A$.

**Proposition 2.4.18** For any subsets $A_1$ and $A_2$ of $(X, \tau)$,

i) $\omega\text{-int}(A_1 \cap A_2) = \omega\text{-int}(A_1) \cap \omega\text{-int}(A_2)$,
ii) \( \omega\text{-int}(A_1 \cup A_2) \supseteq \omega\text{-int}(A_1) \cup \omega\text{-int}(A_2) \),

iii) If \( A_1 \subseteq A_2 \), then \( \omega\text{-int}(A_1) \subseteq \omega\text{-int}(A_2) \),

iv) \( \omega\text{-int}(X) = X \) and

v) \( \omega\text{-int}(\emptyset) = \emptyset \).

**Proposition 2.4.19**

i) \( (\omega\text{-int}(A))^c = \omega\text{-cl}(A^c) \),

ii) \( \omega\text{-int}(A) = (\omega\text{-cl}(A^c))^c \),

iii) \( \omega\text{-cl}(A) = (\omega\text{-int}(A^c))^c \).

**Proof**: i). Let \( x \in (\omega\text{-int}(A))^c \). Then \( x \notin \omega\text{-int}(A) \), i.e., every \( \omega \)-open set \( U \) containing \( x \) is such that \( U \nsubseteq A \). i.e., every \( \omega \)-open set \( U \) containing \( x \) is such that \( U \cap A^c \neq \emptyset \). By Lemma 2.4.14, \( x \in \omega\text{-cl}(A^c) \) and therefore \( (\omega\text{-int}(A))^c \subseteq \omega\text{-cl}(A^c) \).

Conversely, let \( x \in \omega\text{-cl}(A^c) \). Then by Lemma 2.4.14, every \( \omega \)-open set \( U \) containing \( x \) is such that \( U \cap A^c \neq \emptyset \). i.e., every \( \omega \)-open set \( U \) containing \( x \) is such that \( U \nsubseteq A \). This implies by Definition 2.4.15, \( x \notin \omega\text{-int}(A) \). i.e., \( x \in (\omega\text{-int}(A))^c \) and so \( \omega\text{-cl}(A^c) \subseteq (\omega\text{-int}(A))^c \). Thus \( (\omega\text{-int}(A))^c = \omega\text{-cl}(A^c) \).

ii) Follows by taking complements in (i).

iii) Follows by replacing \( A \) by \( A^c \) in (i).

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### 2.5. Applications

As applications of \( \omega \)-closed sets, we introduce four new topological spaces namely, \( T_\omega \) spaces, \( gT_\omega \) spaces, \( \alpha T_\omega \) spaces, and \( wT_\omega \) spaces. Using these spaces we obtain certain new characterizations for the spaces \( T_{1/2} \), \( \alpha T_b \) and \( T_{wT} \).
Levine [59], Devi et al [23] and Nagaveni [79] introduced $T_{1/2}$ spaces, $\alpha T_b$ spaces and $T_{wg}$ spaces respectively. We prove that $T_0$ (resp. $gT_0$) spaces are strictly weaker than $T_{1/2}$, $T_b$ and $\alpha T_b$ (resp. $T_{1/2}$) spaces.

Further, it is shown that $\alpha T_0$ (resp. $wT_0$) is properly placed between $\alpha T_b$ and $\alpha T_d$ (resp. $T_{wg}$ and $gT_0$). Also we show that a space $(X, \tau)$ is $T_{1/2}$ (resp. $aT_b$, $T_{wg}$) if and only if it is both $T_0$ and $gT_0$ (resp. $T_0$ and $\alpha T_0$, $T_0$ and $wT_0$).

First we introduce the following definition.

**Definition 2.5.1** A space $(X, \tau)$ is called a $T_0$ space if every $\omega$-closed set in it is closed. A typical example of a $T_0$ space is the digital plane $(\mathbb{Z}^2, \mathcal{K}^2)$.

Dontchev [30] proved that a space $(X, \tau)$ is $T_{1/2}$ if and only if it is semi-$T_{1/2}$ and semi-pre-$T_{1/2}$. Recently Veera Kumar [121] proved that a space $(X, \tau)$ is $T_{1/2}$ if and only if it is $T_{1/2}$ and $T_{1/2}$. The following proposition shows that the class of $T_0$ spaces properly contains the class of $T_{1/2}$ spaces.

**Proposition 2.5.2** Every $T_{1/2}$ space is $T_0$ but not conversely.

**Proof**: Follows from Proposition 2.2.4.

As an application of $\alpha$-closed sets (resp. gsp-closed sets and $g^*$-closed sets) Njastad [81] (resp. Dontchev [30] and Veera Kumar [121]) introduced $\alpha$-space (resp. semi-pre-$T_{1/2}$ space and $T^{*}_{1/2}$, $T^{*}_{1/2}$ spaces). We prove that $T_0$ space is independent of these spaces.

**Remark 2.5.3** $T_0$ space and $\alpha$-space are independent. The space $(X, \tau)$ of Example 2.2.5 is a $T_0$ space but not an $\alpha$-space and the space $(X, \tau)$ of Example 2.2.3 is an $\alpha$-space but not a $T_0$ space.

**Remark 2.5.4** $T_0$ and semi-pre-$T_{1/2}$ are independent. The space $(X, \tau)$ of Example 2.3.4 is semi-pre $T_{1/2}$ but not $T_0$ and the space $(X, \tau)$ of Example 2.2.5 is $T_0$ but not semi-pre-$T_{1/2}$.

**Remark 2.5.5** $T_0$ and $T^{*}_{1/2}$ are independent as it can be seen from the
following examples.

**Example 2.5.6** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then $(X, \tau)$ is $T_\omega$ but not $T^{*/1/2}$.

**Example 2.5.7** Let $X$ and $\tau$ be as in the Example 2.2.3. Then $(X, \tau)$ is not a $T_\omega$ space, since $\{b\}$ is a $\omega$-closed set but not a closed set of $(X, \tau)$. However, $(X, \tau)$ is a $T^{*/1/2}$ space.

**Remark 2.5.8** $T_\omega$ and $T^{*/1/2}$ are independent. The space $(X, \tau)$ in Example 2.2.5 is $T_\omega$ but not $T^{*/1/2}$ and the space $(X, \tau)$ in Example 2.3.4 is $T^{*/1/2}$ but not $T_\omega$.

The proofs of the following two propositions follows from Proposition 2.2.8.

**Proposition 2.5.9** Every $\alpha T_b$ space is a $T_\omega$ space but not conversely.

The space $(X, \tau)$ of Example 2.2.5 is a $T_\omega$ space but not an $\alpha T_b$ space.

**Proposition 2.5.10** Every $T_b$ space is a $T_\omega$ space but not conversely.

The space $(X, \tau)$ of Example 2.2.5 is a $T_\omega$ space but not a $T_b$ space.

**Theorem 2.5.11** For a space $(X, \tau)$, the following are equivalent:

i) $(X, \tau)$ is a $T_\omega$ space,

ii) Every singleton of $(X, \tau)$ is either semi-closed or open.

**Proof:** i) $\Rightarrow$ ii) : Assume that for some $x \in X$, the set $\{x\}$ is not a semi-closed set of $(X, \tau)$. Then the only semi-open set containing $\{x\}^c$ is $X$ and so $\{x\}^c$ is $\omega$-closed in $(X, \tau)$. By assumption, $\{x\}^c$ is closed or equivalently $\{x\}$ is open in $(X, \tau)$.

ii) $\Rightarrow$ i) : Let $A$ be an $\omega$-closed subset of $(X, \tau)$ and let $x \in \text{cl}(A)$. By assumption, $\{x\}$ is either semi-closed or open.

Case 1 : Suppose $\{x\}$ is semi-closed. If $x \notin A$, then $\text{cl}(A) - A$ contains a non-empty semi-closed set $\{x\}$, which is a contradiction to Theorem 2.2.28. Therefore $x \in A$. 

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Case 2: Suppose \( \{x\} \) is open. Since \( x \in \text{cl}(A) \), \( \{x\} \cap A \neq \emptyset \) and so \( x \in A \). Thus in both cases, \( x \in A \) and therefore \( \text{cl}(A) \subseteq A \) or equivalently \( A \) is a closed set of \((X, \tau)\).

**Definition 2.5.12** A space \((X, \tau)\) is called a \( gT_0 \) space if every \( g \)-closed set of \((X, \tau)\) is an \( \omega \)-closed set.

**Proposition 2.5.13** Every \( T_{1/2} \) space is a \( gT_0 \) space but not conversely.

**Proof:** Follows from Proposition 2.2.2.

The space \((X, \tau)\) of Example 2.2.3 is a \( gT_0 \) space but not a \( T_{1/2} \) space.

We show that \( gT_0 \) is independent of \( T_0 \), \( T_{1/2} \) and \( T_{1/2} \).

**Remark 2.5.14** \( T_0 \) and \( gT_0 \) are independent. The space \((X, \tau)\) of Example 2.2.5 is \( T_0 \) but not \( gT_0 \) and the space \((X, \tau)\) of Example 2.3.4 is \( gT_0 \) but not \( T_0 \).

**Remark 2.5.15** \( gT_0 \) and \( T_{1/2} \) are independent. The space \((X, \tau)\) in Example 2.3.4 is \( gT_0 \) but not \( T_{1/2} \) and the space \((X, \tau)\) in Example 2.2.5 is \( T_{1/2} \) but not \( gT_0 \).

**Remark 2.5.16** \( gT_0 \) and \( T_{1/2} \) are independent. The space \((X, \tau)\) of Example 2.2.3 is \( gT_0 \) but not \( T_{1/2} \) and the space \((X, \tau)\) of Example 2.5.6 is \( T_{1/2} \) but not \( gT_0 \).

**Proposition 2.5.17** If \((X, \tau)\) is a \( gT_0 \) space, then every singleton subset of \((X, \tau)\) is \( g \)-closed or \( \omega \)-open.

**Proof:** Suppose that for some \( x \in X \), the set \( \{x\} \) is not \( g \)-closed. Then \( \{x\} \) is not a closed set, since every closed set is a \( g \)-closed set. So \( \{x\}^c \) is not open and the only open set containing \( \{x\}^c \) is \( X \) itself. Therefore \( \{x\}^c \) is trivially a \( g \)-closed set and by assumption, \( \{x\}^c \) is an \( \omega \)-closed set or equivalently \( \{x\} \) is \( \omega \)-open.
**Remark 2.5.18** The converse of Proposition 2.5.17 is not true in general. The space \((X, \tau)\) of Example 2.2.5 satisfies the conclusion of Proposition 2.5.17, but it is not a \(gT_\omega\) space.

**Theorem 2.5.19** A space \((X, \tau)\) is \(T_{1/2}\) if and only if it is both \(T_\omega\) and \(gT_\omega\).

**Proof:** Necessity follows from Proposition 2.5.2 and Proposition 2.5.13.

Sufficiency: Assume \((X, \tau)\) is both \(T_\omega\) and \(gT_\omega\). Let \(A\) be a \(g\)-closed set of \((X, \tau)\). Then \(A\) is \(\omega\)-closed, since \((X, \tau)\) is a \(gT_\omega\) space. Again since \((X, \tau)\) is a \(T_\omega\) space, \(A\) is a closed set in \((X, \tau)\) and so \((X, \tau)\) is a \(T_{1/2}\) space.

Thus the class of \(T_\omega\) spaces is the dual of the class of \(gT_\omega\) spaces to the class of \(T_{1/2}\) spaces.

**Remark 2.5.20** \([121]\) \(T^{*}_{1/2}\) and \(T^{*}_{1/2}\) are independent.

**Remark 2.5.21** The characterization of \(T_{1/2}\) spaces in Theorem 2.5.19 is different from that of Veera Kumar's \([121]\). This is clear from Remarks 2.5.5, 2.5.8, 2.5.14, 2.5.15, 2.5.16 and 2.5.20.

**Theorem 2.5.22** \([39]\) A topological space \((X, \tau)\) is \(T_{1/2}\) iff for each \(x \in X\), either \(\{x\}\) is open or \(\{x\}\) is closed.

**Theorem 2.5.23** \([21]\) For a topological space \((X, \tau)\), the following are equivalent:

a) Every \(gs\)-closed set of \((X, \tau)\) is semi-closed.

b) For each \(x \in X\), \(\{x\}\) is closed or semi-open in \((X, \tau)\).

c) For each \(x \in X\), \(\{x\}\) is closed or open in \((X, \tau)\).

d) \((X, \tau)\) is a \(T_{1/2}\) space.

**Theorem 2.5.24** \([31]\) For a topological space \((X, \tau)\) the following conditions are equivalent:

(1) \((X, \tau)\) is a \(T_{1/2}\) space.

(2) Every \(\alpha g\)-closed subset of \((X, \tau)\) is \(\alpha\)-closed.

**Theorem 2.5.25** \([30]\) For a topological space \((X, \tau)\) the following conditions are equivalent:
(1) \((X, \tau)\) is a \(T_{1/2}\) space.

(2) \((X, \tau)\) is a semi-\(T_{1/2}\) space and a semi-pre-\(T_{1/2}\) space.

**Theorem 2.5.26**[121] A space \((X, \tau)\) is a \(T_{1/2}\) space iff it is \(T_{1/2}\) and \(T^{*}_{1/2}\).

**Theorem 2.5.27**[34] For a topological space \((X, \tau)\) the following conditions are equivalent:

(1) \((X, \tau)\) is a \(T_{1/2}\) space.

(2) Every \(\emptyset\)-generalized closed set is closed.

**Theorem 2.5.28** For a topological space \((X, \tau)\), the following conditions are equivalent:

i) \((X, \tau)\) is a \(T_{1/2}\) space.

ii) Every singleton of \((X, \tau)\) is either open or closed.

iii) Every singleton of \((X, \tau)\) is either semi-open or closed.

iv) Every \(\alpha g\)-closed set of \((X, \tau)\) is \(\alpha\)-closed.

v) Every \(gs\)-closed set of \((X, \tau)\) is semi-closed.

vi) Every \(\emptyset\)-generalized closed set of \((X, \tau)\) is closed.

vii) \((X, \tau)\) is a semi-\(T_{1/2}\) space and a semi-pre-\(T_{1/2}\) space.

viii) \((X, \tau)\) is a \(T^{*}_{1/2}\) space and \(T_{1/2}\) space.

ix) \((X, \tau)\) is a \(T_\omega\) space and \(gT_\omega\) space.

**Proof:** i) \(\Leftrightarrow\) ii) (Theorem 2.5.22). i) \(\Leftrightarrow\) iii) \(\Leftrightarrow\) v) (Theorem 2.5.23). i) \(\Leftrightarrow\) iv) (Theorem 2.5.24). i) \(\Leftrightarrow\) vii) (Theorem 2.5.25). i) \(\Leftrightarrow\) viii) (Theorem 2.5.26). i) \(\Leftrightarrow\) vi) (Theorem 2.5.27) and i) \(\Leftrightarrow\) ix) follows from Theorem 2.5.19.

We introduce a new topological space called \(\alpha T_\omega\) space as follows:

**Definition 2.5.29** A space \((X, \tau)\) is called an \(\alpha T_\omega\) space if every \(\alpha g\)-closed set of \((X, \tau)\) is \(\omega\)-closed.

**Remark 2.5.30** \(\alpha T_\omega\) is independent of \(T_\omega\). The space \((X, \tau)\) of Example 2.3.4 is \(\alpha T_\omega\) but not \(T_\omega\) and the space \((X, \tau)\) in Example 2.2.5 is \(T_\omega\) but not \(\alpha T_\omega\).
Devi et al [23] introduced $\alpha T_b$ and $\alpha T_d$ spaces and proved that every $\alpha T_b$ space is $\alpha T_d$ and $T_{1/2}$. We prove that $\alpha T_\omega$ spaces are properly placed between $\alpha T_b$ and $\alpha T_d$.

**Proposition 2.5.31** Every $\alpha T_b$ space is an $\alpha T_\omega$ space but not conversely.

**Proof:** Let $(X, \tau)$ be an $\alpha T_b$ space and let $A$ be an $\alpha g$-closed set of $(X, \tau)$. Then $A$ is a closed subset of $(X, \tau)$ and by Proposition 2.2.2, $A$ is $\omega$-closed. Therefore $(X, \tau)$ is an $\alpha T_\omega$ space.

The space $(X, \tau)$ of Example 2.3.4 is an $\alpha T_\omega$ space but not an $\alpha T_b$ space.

**Proposition 2.5.32** Every $\alpha T_\omega$ space is an $\alpha T_d$ space but not conversely.

**Proof:** Follows from Proposition 2.2.4.

The space in Example 2.2.5 is an $\alpha T_d$ space but not an $\alpha T_\omega$ space.

**Proposition 2.5.33** If $(X, \tau)$ is an $\alpha T_\omega$ space, then every singleton subset of $(X, \tau)$ is $\alpha g$-closed or $\omega$-open.

**Proof:** Assume that for some $x \in X$, the set $\{x\}$ is not $\alpha g$-closed. Then $\{x\}$ is not a closed set and so $\{x\}^c$ is not an open set. Thus the only open set containing $\{x\}^c$ is $X$ itself and hence $\{x\}^c$ is trivially an $\alpha g$-closed set. By assumption $\{x\}^c$ is a $\omega$-closed set or equivalently $\{x\}$ is $\omega$-open.

**Remark 2.5.34** The converse of this proposition need not be true.

The space $(X, \tau)$ of Example 2.2.5 is not an $\alpha T_\omega$ space. However it satisfies the conclusion of Proposition 2.5.33.

**Theorem 2.5.35** A space $(X, \tau)$ is $\alpha T_b$ if and only if it is both $\alpha T_\omega$ and $T_\omega$.

**Proof:** Necessity follows from Proposition 2.5.9 and Proposition 2.5.31. Sufficiency: Assume $(X, \tau)$ is both $\alpha T_\omega$ and $T_\omega$. Let $A$ be an $\alpha g$-closed set of $(X, \tau)$. Since $(X, \tau)$ is an $\alpha T_\omega$ space, $A$ is $\omega$-closed in $(X, \tau)$. Since $(X, \tau)$ is also a $T_\omega$ space, $A$ is closed and so $(X, \tau)$ is an $\alpha T_b$ space.
**Definition 2.5.36** A space \((X, \tau)\) is called a \(wg\, T_\omega\) space if every \(wg\)-closed set of \((X, \tau)\) is \(\omega\)-closed.

Nagaveni [79] introduced \(Twg\) spaces and proved that every \(Twg\) is an \(\alpha\)-space, a semi-pre-\(T_{1/2}\) space and a \(T_\omega\) space but not conversely. In the next two propositions we prove that \(wg\, T_\omega\) is properly placed between \(Twg\) and \(g\, T_\omega\).

**Proposition 2.5.37** Every \(Twg\) space is \(wg\, T_\omega\) but not conversely.

**Proof:** Follows from Proposition 2.2.2.

The space \((X, \tau)\) of Example 2.2.3 is a \(wg\, Tw\) space but not a \(Twg\) space.

**Theorem 2.5.38 [79]** If a subset \(A\) of a topological space \((X, \tau)\) is \(g\)-closed then it is \(wg\)-closed in \((X, \tau)\) but not conversely.

**Proposition 2.5.39** Every \(wg\, T_\omega\) space is a \(g\, T_\omega\) space but not conversely.

**Proof:** Follows from Theorem 2.5.38.

The space \((X, \tau)\) of Example 2.3.4 is \(g\, T_\omega\) but not \(wg\, T_\omega\).

**Remark 2.5.40** The spaces \(T_\omega\) and \(wg\, T_\omega\) are independent. The space \((X, \tau)\) of Example 2.2.5 is \(T_\omega\) but not \(wg\, T_\omega\) and the space \((X, \tau)\) of Example 2.2.3 is \(wg\, T_\omega\) but not \(T_\omega\).

**Proposition 2.5.41** If \((X, \tau)\) is a \(wg\, T_\omega\) space, then for each \(x \in X\), \(\{x\}\) is either \(wg\)-closed or \(\omega\)-open.

**Proof:** Assume that \((X, \tau)\) is a \(wg\, T_\omega\) space and for some \(x \in X\), \(\{x\}\) is not a \(wg\)-closed set. Then \(\{x\}\) is not a closed set, since every closed set is \(wg\)-closed. So \(\{x\}^c\) is not an open set and \(\{x\}^c\) is a \(g\)-closed set, since \(X\) is the only open set which contains \(\{x\}^c\). By Theorem 2.5.38, \(\{x\}^c\) is \(wg\)-closed and by assumption \(\{x\}^c\) is \(\omega\)-closed or equivalently \(\{x\}\) is \(\omega\)-open.

**Remark 2.5.42** The converse of Proposition 2.5.41 is not true in general. The space \((X, \tau)\) of Example 2.3.4 satisfies the conclusion of the Proposition.
2.5.41, but it is not a $\text{wg}T_\omega$ space.

**Proposition 2.5.43** Every singleton set in a $T_{\text{wg}}$ space $(X, \tau)$ is either $\text{wg}$-closed or open.

**Proof:** Assume that $(X, \tau)$ is a $T_{\text{wg}}$ space and for some $x \in X$, $\{x\}$ is not $\text{wg}$-closed. Then $\{x\}$ is not closed, since every closed set is $\text{wg}$-closed. So $\{x\}^c$ is not open and hence $\{x\}^c$ is $\text{wg}$-closed, since the only open set containing $\{x\}^c$ is $X$ itself. By assumption $\{x\}^c$ is closed or equivalently $\{x\}$ is open in $(X, \tau)$.

**Remark 2.5.44** The converse of this proposition is not true in general.

The space $(X, \tau)$ of Example 2.2.3 is not a $T_{\text{wg}}$ space, but it satisfies the conclusion of the Proposition 2.5.43.

**Theorem 2.5.45** Every $T_{\text{wg}}$ space is $T_\omega$ space but not conversely.

**Theorem 2.5.46** A space $(X, \tau)$ is $T_{\text{wg}}$ if and only if is both $T_\omega$ and $\text{wg}T_\omega$.

**Proof:** Necessity follows from Proposition 2.5.37 and Theorem 2.5.45. Sufficiency: Suppose $(X, \tau)$ is both $T_\omega$ and $\text{wg}T_\omega$. Let $A$ be a $\text{wg}$-closed set of $(X, \tau)$. Then $A$ is $\omega$-closed, since $(X, \tau)$ is $\text{wg}T_\omega$. Again by assumption, $A$ is closed and so $(X, \tau)$ is $T_{\text{wg}}$.

**Remark 2.5.47** The spaces $\text{wg}T_\omega$ and $T_{1/2}$ are independent. The space $(X, \tau)$ of Example 2.2.3 is $\text{wg}T_\omega$ but not $T_{1/2}$ and the space $(X, \tau)$ in Example 2.5.6 is $T_{1/2}$ but not $\text{wg}T_\omega$. 

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Remark 2.5.48 We conclude this section with the following diagram which results from the above discussions and known results:

\[ \begin{align*}
\alpha T_b & \rightarrow \alpha \text{-space} \rightarrow \text{semi-pre-} T_{1/2} \leftarrow T_{wg} \\
\alpha T_\omega & \rightarrow T_\omega \leftarrow T_{1/2} \\
\alpha T_d & \rightarrow T_{1/2} \rightarrow T_\omega \\
T_b & \rightarrow T_{1/2} \rightarrow T_\omega \\
\end{align*} \]

Remark 2.5.49 A topological space \((X, \tau)\) is \(T_\omega\) if and only if \((X, \tau)\) is semi-\(T_{1/2}\).