CHAPTER IV

\(\omega\)-HOMEOMORPHISMS IN TOPOLOGICAL SPACES

4.1. Introduction


In this chapter we first introduce \(\omega\)-closed maps in topological spaces and then we introduce and study \(\omega\)-homeomorphisms which are weaker than homeomorphisms. We prove that gc-homeomorphism and \(\omega\)-homeomorphism are independent. We also introduce \(\omega^*\)-homeomorphisms and prove that the set of all \(\omega^*\)-homeomorphisms forms a group under the operation composition of maps. In this chapter, we further introduce and study the new concepts namely \(\omega\)-compactness, \(\omega\)-connectedness, \(\omega\)-regular spaces and \(\omega\)-normal spaces in topological spaces.

4.2. \(\omega\)-closed maps

Malghan [71] introduced the concept of generalized closed maps in topological spaces. Devi [21] introduced and studied sg-closed maps gs-closed maps. Recently, Gnanambal [48] defined gpr-closed maps and studied some of their properties. In this section, we introduce \(\omega\)-closed
Definition 4.2.1 A map \( f: (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( \omega \)-closed if the image of every closed set in \( (X, \tau) \) is \( \omega \)-closed in \( (Y, \sigma) \).

Example 4.2.2 Let \( X = Y = \{ a, b, c \} \), \( \tau = \{ \emptyset, \{ a \}, \{ b \}, \{ a, b \}, X \} \) and \( \sigma = \{ \emptyset, \{ a \}, \{ b, c \}, Y \} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. Then \( f \) is an \( \omega \)-closed map.

Proposition 4.2.3 A mapping \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( \omega \)-closed if and only if \( \omega\text{-cl}(f(A)) \subseteq f(\text{cl}(A)) \) for every subset \( A \) of \( (X, \tau) \).

Proof: Suppose that \( f \) is \( \omega \)-closed and \( A \subseteq X \). Then \( f(\text{cl}(A)) \) is \( \omega \)-closed in \( (Y, \sigma) \). We have \( f(A) \subseteq f(\text{cl}(A)) \) and by Propositions 2.3.9 and 2.3.10, \( \omega\text{-cl}(f(A)) \subseteq \omega\text{-cl}(f(\text{cl}(A))) = f(\text{cl}(A)) \).

Conversely, let \( A \) be any closed set in \( (X, \tau) \). Then \( A = \text{cl}(A) \) and so \( f(A) = f(\text{cl}(A)) \supseteq \omega\text{-cl}(f(A)) \), by hypothesis. We have \( f(A) \subseteq \omega\text{-cl}(f(A)) \) by Proposition 2.3.9. Therefore \( f(A) = \omega\text{-cl}(f(A)) \). i.e., \( f(A) \) is \( \omega \)-closed by Proposition 2.3.9 and hence \( f \) is \( \omega \)-closed.

Proposition 4.2.4 Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function such that \( \omega\text{-cl}(f(A)) \subseteq f(\text{cl}(A)) \) for every subset \( A \subseteq X \). Then the image \( f(A) \) of a closed set \( A \) in \( (X, \tau) \) is \( \tau_{\omega} \)-closed in \( (Y, \sigma) \).

Proof: Let \( A \) be a closed set in \( (X, \tau) \). Then by hypothesis \( \omega\text{-cl}(f(A)) \subseteq f(\text{cl}(A)) = f(A) \) and so \( \omega\text{-cl}(f(A)) = f(A) \). Therefore \( f(A) \) is \( \tau_{\omega} \)-closed in \( (Y, \sigma) \).

Proposition 4.2.5 If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is an \( \omega \)-closed mapping then for each subset \( A \) of \( (X, \tau) \), \( \text{cl}(\text{int}(f(A))) \subseteq f(\text{cl}(A)) \).

Proof: Let \( f \) be an \( \omega \)-closed map and \( A \subseteq X \). Then since \( \text{cl}(A) \) is a closed set in \( (X, \tau) \), we have \( f(\text{cl}(A)) \) is \( \omega \)-closed and hence pre-closed by
Proposition 2.2.7. Therefore, cl(int(f(cl(A)))) ⊆ f(cl(A)). i.e., cl(int(f(A)) ⊆ f(cl(A)).

The converse of this proposition need not be true in general as seen from the following example.

Example 4.2.6 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then for each subset $A \subseteq X$, we have cl(int(f(A))) ⊆ f(cl(A)), but $f$ is not an $\omega$-closed map.

Theorem 4.2.7 A map $f: (X, x) \rightarrow (Y, a)$ is $\omega$-closed if and only if for each subset $S$ of $(Y, \sigma)$ and for each open set $U$ containing $f^{-1}(S)$ there is an $\omega$-open set $V$ of $(Y, \sigma)$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof: Suppose that $f$ is $\omega$-closed. Let $S \subseteq Y$ and $U$ be open set of $(X, \tau)$ such that $f^{-1}(S) \subseteq U$. Then $V = (f(U^c))^c$ is an $\omega$-open set containing $S$ such that $f^{-1}(V) \subseteq U$.

For the converse, let $F$ be a closed set of $(X, \tau)$. Then $f^{-1}((f(F))^c) \subseteq F^c$ and $F^c$ is open. By assumption, there exists an $\omega$-open set $V$ of $(Y, \sigma)$ such that $(f(F))^c \subseteq V$ and $f^{-1}(V) \subseteq F^c$ and so $F \subseteq (f^{-1}(V))^c$. Hence $V^c \subseteq f(F) \subseteq f((f^{-1}(V))^c) \subseteq V^c$ which implies $f(F) = V^c$. Since $V^c$ is $\omega$-closed, $f(F)$ is $\omega$-closed and therefore $f$ is $\omega$-closed.

Proposition 4.2.8 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is irresolute $\omega$-closed and $A$ is an $\omega$-closed subset of $(X, \tau)$, then $f(A)$ is $\omega$-closed.

Proof: Let $U$ be a semi-open set in $(Y, \sigma)$ such that $f(A) \subseteq U$. Since $f$ is irresolute, $f^{-1}(U)$ is a semi-open set containing $A$. Hence $cl(A) \subseteq f^{-1}(U)$ as $A$ is $\omega$-closed in $(X, \tau)$. Since $f$ is $\omega$-closed, $f(cl(A))$ is an $\omega$-closed set contained in the semi-open set $U$, which implies that $cl(f(cl(A)) \subseteq U$ and hence $cl(f(A)) \subseteq U$. Therefore, $f(A)$ is an $\omega$-closed set.

The following example shows that the composition of two $\omega$-closed maps need not be $\omega$-closed.
**Example 4.2.9** Let $(Y, \sigma)$ and $f$ be as in Example 4.2.2. Let $Z = \{a, b, c\}$ and $\eta = \{\phi, \{a, c\}, Z\}$. Define, $g : (Y, \sigma) \to (Z, \eta)$ by $g(a) = g(b) = b$ and $g(c) = a$. Then both $f$ and $g$ are $\omega$-closed maps but their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is not an $\omega$-closed map, since for the closed set $\{c\}$ in $(X, \tau)$, $(g \circ f)(\{c\}) = \{a\}$, which is not an $\omega$-closed set in $(Z, \eta)$.

**Corollary 4.2.10** Let $f : (X, \tau) \to (Y, \sigma)$ be $\omega$-closed and $g : (Y, \sigma) \to (Z, \eta)$ be $\omega$-closed and irresolute, then their composition $g \circ f : (X, \tau) \to (Z, \sigma)$ is $\omega$-closed.

**Proof:** Let $A$ be a closed set of $(X, \tau)$. Then by hypothesis $f(A)$ is an $\omega$-closed set in $(Y, \sigma)$. Since $g$ is both $\omega$-closed and irresolute by Proposition 4.2.8, $g(f(A)) = (g \circ f)(A)$ is $\omega$-closed in $(Z, \eta)$ and therefore $g \circ f$ is $\omega$-closed.

**Proposition 4.2.11** Let $f : (X, \tau) \to (Y, \sigma)$, $g : (Y, \sigma) \to (Z, \eta)$ be $\omega$-closed maps and $(Y, \sigma)$ be a $T_\omega$ space. Then their composition $g \circ f : (X, \tau) \to (Z, \sigma)$ is $\omega$-closed.

**Proof:** Let $A$ be a closed set of $(X, \tau)$. Then by assumption $f(A)$ is $\omega$-closed in $(Y, \sigma)$. Since $(Y, \sigma)$ is a $T_\omega$ space, $f(A)$ is closed in $(Y, \sigma)$ and again by assumption $g(f(A))$ is $\omega$-closed in $(Z, \eta)$. i.e., $(g \circ f)(A)$ is $\omega$-closed in $(Z, \eta)$ and so $g \circ f$ is $\omega$-closed.

**Proposition 4.2.12** If $f : (X, \tau) \to (Y, \sigma)$ is $\omega$-closed, $g : (Y, \sigma) \to (Z, \tau)$ is $g$-closed (resp. pre closed, rg-closed and gp-closed) and $(Y, \sigma)$ is a $T_\omega$ space, then their composition $g \circ f : (X, \tau) \to (Z, \sigma)$ is $g$-closed (resp. preclosed, rg-closed, and gp-closed).

**Proof:** Similar to Proposition 4.2.11.

**Proposition 4.2.13** Let $f : (X, \tau) \to (Y, \sigma)$ be a closed map and $g : (Y, \sigma) \to (Z, \eta)$ be an $\omega$-closed map, then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is $\omega$-closed.

**Proof:** Similar to Proposition 4.2.11.
**Remark 4.2.14** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \omega \)-closed and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is closed, then their composition need not be an \( \omega \)-closed map as seen from the following example.

**Example 4.2.15** Let \((Y, \sigma)\) and \(f\) be as in Example 4.2.2. Let \(Z = \{a, b, c\}\) and \(\eta = \{\phi, \{b\}, \{c\}, \{b, c\}, Z\}\). Define \(g : (Y, \sigma) \rightarrow (Z, \eta)\) by \(g(a) = g(b) = a\) and \(g(c) = b\). Then \(f\) is an \( \omega \)-closed map and \(g\) is a closed map. But their composition \(g \circ f : (X, \tau) \rightarrow (Z, \eta)\) is not an \( \omega \)-closed map, since for the closed set \(\{c\}\) in \((X, \tau)\), \((g \circ f)(\{c\}) = \{b\}\), which is not \( \omega \) closed in \((Z, \eta)\).

**Theorem 4.2.16** Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) and \(g : (Y, \sigma) \rightarrow (Z, \eta)\) be two mappings such that their composition \(g \circ f : (X, \tau) \rightarrow (Y, \sigma)\) be an \( \omega \)-closed mapping. Then the following statements are true.

i) If \(f\) is continuous and surjective, then \(g\) is \( \omega \)-closed.

ii) If \(g\) is \( \omega \)-irresolute and injective, then \(f\) is \( \omega \)-closed.

iii) If \(f\) is \(g\)-continuous, surjective and \((X, \tau)\) is a \(T_{1/2}\) space then \(g\) is \( \omega \)-closed.

iv) If \(g\) is strongly \( \omega \)-continuous and injective, then \(f\) is closed.

**Proof:**

i). Let \(A\) be a closed set of \((Y, \sigma)\). Since \(f\) is continuous, \(f^{-1}(A)\) is closed in \((X, \tau)\) and since \(g \circ f\) is \( \omega \)-closed, \((g \circ f)(f^{-1}(A))\) is \( \omega \)-closed in \((Z, \sigma)\). i.e., \(g(A)\) is \( \omega \)-closed in \((Z, \sigma)\), since \(f\) is surjective. Therefore \(g\) is an \( \omega \)-closed map.

ii). Let \(B\) be a closed set of \((X, \tau)\). Since \(g \circ f\) is \( \omega \)-closed, \((g \circ f)(B)\) is \( \omega \)-closed in \((Z, \eta)\). Since \(g\) is \( \omega \)-irresolute, \(g^{-1}((g \circ f)(B))\) is \( \omega \)-closed in \((Y, \sigma)\). i.e., \(f(B)\) is \( \omega \)-closed in \((Y, \sigma)\), since \(g\) is injective. Thus \(f\) is an \( \omega \)-closed map.

iii). Let \(C\) be a closed set of \((Y, \sigma)\). Since \(f\) is \(g\)-continuous, \(f^{-1}(A)\) is \(g\)-closed in \((X, \tau)\). Since \((X, \tau)\) is a \(T_{1/2}\) space, \(f^{-1}(A)\) is closed in \((X, \tau)\) and so as in i), \(g\) is an \( \omega \)-closed map.
iv). Let $D$ be a closed set of $(X, \tau)$. Since $g \circ f$ is $\omega$-closed, $(g \circ f)(D)$ is $\omega$-closed in $(Z, \eta)$. Since $g$ is strongly $\omega$-continuous, $g^{-1}((g \circ f)(D))$ is closed in $(Y, \sigma)$. i.e., $f(D)$ is closed in $(Y, \sigma)$, since $g$ is injective. Therefore $f$ is a closed map.

Regarding the restriction $f_A$ of a map $f : (X, \tau) \to (Y, \sigma)$ to a subset $A$ of $(X, \tau)$, we have the following:

**Theorem 4.2.17** Let $(X, \tau)$ and $(Y, \sigma)$ be any topological spaces. Then

i) If $f : (X, \tau) \to (Y, \sigma)$ is $\omega$-closed and $A$ is a closed subset of $(X, \tau)$, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is $\omega$-closed.

ii) If $f : (X, \tau) \to (Y, \sigma)$ is irresolute and $\omega$-closed and $A$ is an open and $\omega$-closed set of $(X, \tau)$, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is $\omega$-closed.

iii) If $f : (X, \tau) \to (Y, \sigma)$ is $\omega$-closed (resp. closed) and $A = f^{-1}(B)$ for some closed (resp. $\omega$-closed) set $B$ of $(Y, \sigma)$, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is $\omega$-closed.

**Proof:**

i). Let $B$ be a closed set of $A$. Then $B = A \cap F$ for some closed set $F$ of $(X, \tau)$ and so $B$ is closed in $(X, \tau)$. By hypothesis, $f(B)$ is $\omega$-closed in $(Y, \sigma)$. But $f(B) = f_A(B)$ and therefore $f_A$ is an $\omega$-closed map.

ii). Let $C$ be a closed set in $A$. Then $C$ is $\omega$-closed in $A$. Since $A$ is both open and $\omega$-closed, $C$ is $\omega$-closed in $(X, \tau)$, by Proposition 2.2.35. Since $f$ is both irresolute and $\omega$-closed, $f(C)$ is $\omega$-closed in $(Y, \sigma)$ by Proposition 4.2.8. Since $f(C) = f_A(C)$, $f_A$ is an $\omega$-closed map.

iii). Let $D$ be a closed set of $A$. Then $D = A \cap H$ for some closed set $H$ in $(X, \tau)$. Now $f_A(D) = f(D) = f(A \cap H) = f(f^{-1}(B) \cap H) = B \cap f(H)$. Since $f$ is $\omega$-closed, $f(H)$ is $\omega$-closed in $(Y, \sigma)$ and so $B \cap f(H)$ is $\omega$-closed in $(Y, \sigma)$ by Corollary 2.2.21. Therefore, $f_A$ is an $\omega$-closed map.

In the next theorem we show that normality is preserved under continuous $\omega$-closed maps.
Theorem 4.2.18 If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a continuous, \( \omega \)-closed map from a normal space \((X, \tau)\) onto a space \((Y, \sigma)\), then \((Y, \sigma)\) is normal.

Proof: Let \( A \) and \( B \) be two disjoint closed subsets of \((Y, \sigma)\). Since \( f \) is continuous, \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint closed sets of \((X, \tau)\). Since \((X, \tau)\) is normal, there exist disjoint open sets \( U \) and \( V \) of \((X, \tau)\) such that \( f^{-1}(A) \subseteq U \) and \( f^{-1}(B) \subseteq V \). Since \( f \) is \( \omega \)-closed, by Theorem 4.2.7, there exist disjoint \( \omega \)-open sets \( G \) and \( H \) in \((Y, \sigma)\) such that \( A \subseteq G \), \( B \subseteq H \), \( f^{-1}(G) \subseteq U \) and \( f^{-1}(H) \subseteq V \). Since \( U \) and \( V \) are disjoint, \( \text{int}(G) \) and \( \text{int}(H) \) are disjoint open sets in \((Y, \sigma)\). Since \( A \) is closed, \( A \) is semi-closed and therefore we have by Proposition 2.4.6, \( A \subseteq \text{int}(G) \). Similarly \( B \subseteq \text{int}(H) \) and hence \((Y, \sigma)\) is normal.

Analogous to an \( \omega \)-closed map, we define an \( \omega \)-open map as follows:

Definition 4.2.19 A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be an \( \omega \)-open map if the image \( f(A) \) is \( \omega \)-open in \((Y, \sigma)\) for each open set \( A \) in \((X, \tau)\).

Proposition 4.2.20 For any bijection \( f : (X, \tau) \rightarrow (Y, \sigma) \), the following statements are equivalent:

i) \( f^{-1} : (Y, \sigma) \rightarrow (X, \tau) \) is \( \omega \)-continuous,

ii) \( f \) is an \( \omega \)-open map and

iii) \( f \) is an \( \omega \)-closed map.

Proof: i) \( \Rightarrow \) ii) : Let \( U \) be an open set of \((X, \tau)\). By assumption \((f^{-1})^{-1}(U) = f(U)\) is \( \omega \)-open in \((Y, \sigma)\) and so \( f \) is \( \omega \)-open.

ii) \( \Rightarrow \) iii) : Let \( F \) be a closed set of \((X, \tau)\). Then \( F^c \) is open in \((X, \tau)\). By assumption, \( f(F^c) \) is \( \omega \)-open in \((Y, \sigma)\). i.e., \( f(F^c) = (f(F))^c \) is \( \omega \)-open in \((Y, \sigma)\) and therefore \( f(F) \) is \( \omega \)-closed in \((Y, \sigma)\). Hence \( f \) is \( \omega \)-closed.

iii) \( \Rightarrow \) i) : Let \( F \) be a closed set in \((X, \tau)\). By assumption \( f(F) \) is \( \omega \)-closed in \((Y, \sigma)\). But \( f(F) = (f^{-1})^{-1}(F) \) and therefore \( f^{-1} \) is \( \omega \)-continuous.

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In the next two theorems, we obtain various characterizations of \( \omega \)-open maps.

**Theorem 4.2.21** Let \( f : (X, \tau) \to (Y, \sigma) \) be a mapping. Then the following statements are equivalent:

i) \( f \) is an \( \omega \)-open mapping.

ii) For a subset \( A \) of \( (X, \tau) \), \( f(\text{int}(A)) \subseteq \omega\text{-int}(f(A)) \).

iii) For each \( x \in X \) and for each neighbourhood \( U \) of \( x \) in \( (X, \tau) \), there exists an \( \omega \)-neighbourhood \( W \) of \( f(x) \) in \( (Y, \sigma) \) such that \( W \subseteq f(U) \).

**Proof:**  
i) \( \Rightarrow \) ii): Suppose \( f \) is \( \omega \)-open. Let \( A \subseteq X \). Then \( \text{int}(A) \) is open in \( (X, \tau) \) and so \( f(\text{int}(A)) \) is \( \omega \)-open in \( (Y, \sigma) \). We have \( f(\text{int}(A)) \subseteq f(A) \). Therefore, by Proposition 2.4.17, \( f(\text{int}(A)) \subseteq \omega\text{-int}(f(A)) \).

ii) \( \Rightarrow \) iii): Suppose ii) holds. Let \( x \in X \) and \( U \) be an arbitrary neighbourhood of \( x \) in \( (X, \tau) \). Then there exists an open set \( G \) such that \( x \in G \subseteq U \). By assumption, \( f(G) = f(\text{int}(G)) \subseteq \omega\text{-int}(f(G)) \). This implies \( f(G) = \omega\text{-int}(f(G)) \). By Proposition 2.4.17, we have \( f(G) \) is \( \omega \)-open in \( (Y, \sigma) \). Further, \( f(x) \in f(G) \subseteq f(U) \) and so iii) holds, by taking \( W = f(G) \).

iii) \( \Rightarrow \) i): Suppose iii) holds. Let \( U \) be any open set in \( (X, \tau) \), \( x \in U \) and \( f(x) = y \). Then \( y \in f(U) \) and for each \( y \in f(U) \), by assumption there exists an \( \omega \)-neighbourhood \( W_y \) of \( y \) in \( (Y, \sigma) \) such that \( W_y \subseteq f(U) \). Since \( W_y \) is an \( \omega \)-neighbourhood of \( y \), there exists an \( \omega \)-open set \( V_y \) in \( (Y, \sigma) \) such that \( y \in V_y \subseteq W_y \). Therefore, \( f(U) = \bigcup \{ V_y : y \in f(U) \} \) is an \( \omega \)-open set in \( (Y, \sigma) \) by Proposition 2.4.4. Thus \( f \) is an \( \omega \)-open mapping.

**Theorem 4.2.22** A function \( f : (X, \tau) \to (Y, \sigma) \) is \( \omega \)-open if and only if for any subset \( S \) of \( (Y, \sigma) \) and for any closed set \( F \) containing \( f^{-1}(S) \), there exists an \( \omega \)-closed set \( K \) of \( (Y, \sigma) \) containing \( S \) such that \( f^{-1}(K) \subseteq F \).

**Proof:** Similar to Theorem 4.2.7.
**Corollary 4.2.23** A function \( f: (X, \tau) \to (Y, \sigma) \) is \( \omega \)-open if and only if \( f^{-1}(\omega\text{-cl}(B)) \subseteq \text{cl}(f^{-1}(B)) \) for each subset \( B \) of \( (Y, \sigma) \).

**Proof:** Suppose that \( f \) is \( \omega \)-open. Then for any \( B \subseteq Y \), \( f^{-1}(B) \subseteq \text{cl}(f^{-1}(B)) \). By Theorem 4.2.22, there exists an \( \omega \)-closed set \( K \) of \( (Y, \sigma) \) such that \( B \subseteq K \) and \( f^{-1}(K) \subseteq \text{cl}(f^{-1}(B)) \). Therefore, \( f^{-1}(\omega\text{-cl}(B)) \subseteq f^{-1}(K) \subseteq \text{cl}(f^{-1}(B)) \), since \( K \) is an \( \omega \)-closed set in \( (Y, \sigma) \).

Conversely, let \( S \) be any subset of \( (Y, \sigma) \) and \( F \) be any closed set containing \( f^{-1}(S) \). Put \( K = \omega\text{-cl}(S) \). Then \( K \) is an \( \omega \)-closed set and \( S \subseteq K \). By assumption, \( f^{-1}(K) = f^{-1}(\omega\text{-cl}(S)) \subseteq \text{cl}(f^{-1}(S)) \subseteq F \) and therefore by Theorem 4.2.22, \( f \) is \( \omega \)-open.

Finally in this section, we define another new class of maps called \( \omega^* \)-closed maps which are stronger than \( \omega \)-closed maps.

**Definition 4.2.24** A map \( f: (X, \tau) \to (Y, \sigma) \) is said to be an \( \omega^* \)-closed map if the image \( f(A) \) is \( \omega \)-closed in \( (Y, \sigma) \) for every \( \omega \)-closed set \( A \) in \( (X, \tau) \).

For example the map \( f \) in Example 4.2.2 is an \( \omega^* \)-closed map.

**Remark 4.2.25** Since every closed set is an \( \omega \)-closed set we have every \( \omega^* \)-closed map is an \( \omega \)-closed map. The converse is not true in general as seen from the following example.

**Example 4.2.26** Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, \{a, b\}, X\} \), \( \sigma = \{\phi, \{a\}, \{a, b\}, \{a, b\}, Y\} \) and \( f: (X, \tau) \to (Y, \sigma) \) be the identity map. Then \( f \) is an \( \omega \)-closed map but not an \( \omega^* \)-closed map, since \( \{a, c\} \) is an \( \omega \)-closed set in \( (X, \tau) \), but its image under \( f \) is \( \{a, c\} \), which is not \( \omega \)-closed in \( (Y, \sigma) \).

**Proposition 4.2.27** A mapping \( f: (X, \tau) \to (Y, \sigma) \), is \( \omega^* \)-closed if and only if \( \omega\text{-cl}(f(A)) \subseteq f(\omega\text{-cl}(A)) \) for every subset \( A \) of \( (X, \tau) \).

**Proof:** Similar to Proposition 4.2.3.

Analogous to \( \omega^* \)-closed map we can also define \( \omega^* \)-open map.
**Proposition 4.2.28** For any bijection $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:

i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\omega$-irresolute.

ii) $f$ is an $\omega^*$-open and

iii) $f$ is an $\omega^*$-closed map.

**Proof:** Similar to Proposition 4.2.20.

**Proposition 4.2.29** If $f : (X, \tau) \to (Y, \sigma)$ is irresolute and $\omega$-closed, then it is an $\omega^*$-closed map.

The proof follows from Proposition 4.2.8.

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### 4.3. $\omega$-Homeomorphisms

In this section we introduce and study two new homeomorphisms namely $\omega$-homeomorphism and $\omega^*$-homeomorphism. We prove that gc-homeomorphism and $\omega$-homeomorphism are independent and $\omega^*$-homeomorphism is an equivalence relation between topological spaces.

**Definition 4.3.1** A bijection $f : (X, \tau) \to (Y, \sigma)$ is called $\omega$-homeomorphism if $f$ is both $\omega$-continuous and $\omega$-open.

**Example 4.3.2** Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then $f$ is bijective, $\omega$-continuous and $\omega$-open and so $f$ is an $\omega$-homeomorphism.

**Proposition 4.3.3** Every homeomorphism is an $\omega$-homeomorphism but not conversely.

The proof is easy consequences of definitions.
The map \( f \) in Example 4.3.2 is an \( \omega \)-homeomorphism but not a homeomorphism because it is not continuous.

**Proposition 4.3.4** Every \( \omega \)-homeomorphism is a \( g \)-homeomorphism but not conversely.

**Proof:** Since every \( \omega \)-continuous map is \( g \)-continuous and every \( \omega \)-open map is \( g \)-open, the proposition follows.

**Example 4.3.5** Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, Y\} \).

Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = c, f(b) = a \) and \( f(c) = b \). Then \( f \) is a \( g \)-homeomorphism but not an \( \omega \)-homeomorphism.

**Proposition 4.3.6** Let \( f : (X, \tau) \to (Y, \sigma) \) be a bijective \( \omega \)-continuous map. Then the following are equivalent:

i) \( f \) is an \( \omega \)-open map,

ii) \( f \) is an \( \omega \)-homeomorphism,

iii) \( f \) is an \( \omega \)-closed map.

**Proof:** Follows from Proposition 4.2.20.

The composition of two \( \omega \)-homeomorphisms need not be an \( \omega \)-homeomorphism as seen from the following example.

**Example 4.3.7** Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \), \( \sigma = \{\emptyset, \{a, b\}, Y\} \) and \( \eta = \{\emptyset, \{a\}, \{a, b\}, Z\} \) respectively. Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be identity maps respectively. Then both \( f \) and \( g \) are \( \omega \)-homeomorphisms but their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is not an \( \omega \)-homeomorphism, because for the open set \( \{b\} \) in \( (X, \tau) \), \((g \circ f)(\{b\}) = \{b\}\), which is not an \( \omega \)-open set in \( (Z, \eta) \). Therefore \( g \circ f \) is not an \( \omega \)-open map and so \( g \circ f \) is not an \( \omega \)-homeomorphism.

We next introduce a new class of maps called \( \omega^* \)-homeomorphisms which forms a sub class of \( \omega \)-homeomorphisms. This class of maps is closed under composition of maps.

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**Definition 4.3.8** A bijection $f : (X, \tau) \to (Y, \sigma)$ is said to be $\omega^*$-homeomorphism if both $f$ and $f^{-1}$ are $\omega$-irresolute.

For Example, the map $f$ in Example 4.3.7 and the maps $h$ and $t$ in Theorem 3.4.25 are $\omega^*$-homeomorphisms.

We denote the family of all $\omega$-homeomorphisms (resp. $\omega^*$-homeomorphism and homeomorphisms) of a topological space $(X, \tau)$ onto itself by $\omega-h(X,\tau)$ (resp. $\omega^*-h(X,\tau)$ and $h(X,\tau)$).

**Proposition 4.3.9** Every $\omega^*$-homeomorphism is an $\omega$-homeomorphism but not conversely. i.e., for any space $(X, \tau)$, $\omega^*-h(X,\tau) \subseteq \omega-h(X,\tau)$.

**Proof:** Follows from Proposition 3.3.6 and every $\omega^*$-open map is $\omega$-open.

The function $g$ in Example 4.3.7 is an $\omega$-homeomorphism but not an $\omega^*$-homeomorphism, since for the $\omega$-closed set $\{a, c\}$ in $(Y, \sigma)$,

$$(g^{-1})^{-1}(\{a, c\}) = g(\{a, c\}) = \{a, c\}$$

which is not $\omega$-closed in $(Z, \eta)$. Therefore $g^{-1}$ is not $\omega$-irresolute and so $g$ is not an $\omega^*$-homeomorphism.

**Theorem 4.3.10 [19]** If $f : (X, \tau) \to (Y, \sigma)$ is continuous and open, then $f$ is irresolute and pre-semi-open.

**Proposition 4.3.11** Every homeomorphism is an $\omega^*$-homeomorphism but not conversely.

**Proof:** Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism. Then $f$ is bijective, continuous and open. By Theorem 4.3.10, $f$ is pre-semi-open and irresolute. Since $f$ is continuous, it is $\omega$-continuous. Thus $f$ is bijective, pre-semi-open and $\omega$-continuous and so $f$ is $\omega$-irresolute by Proposition 3.3.10. Again since $f$ is a homeomorphism it is both continuous and closed. Thus $f$ is bijective, irresolute and closed and therefore, $f^{-1}$ is $\omega$-irresolute by Proposition 3.3.15. Since both $f$ and $f^{-1}$ are $\omega$-irresolute, $f$ is an $\omega^*$-homeomorphism.

The map $f$ in Example 4.3.7 is $\omega^*$-homeomorphism but not a homeomorphism.
**Proposition 4.3.12** Every $\omega^*$-homeomorphism is a $g$-homeomorphism but not conversely.

**Proof:** Follows from Propositions 4.3.9 and 4.3.4.

The map $f$ in Example 4.3.5 is a $g$-homeomorphism but not an $\omega^*$-homeomorphism.

**Remark 4.3.13** $\omega$-homeomorphism and gc-homeomorphism are independent as seen from the following examples.

**Example 4.3.14** Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{b\}, \{b, c\}, X\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then $f$ is a gc-homeomorphism but not an $\omega$-homeomorphism, because $f$ is not an $\omega$-continuous map.

The map $f$ in Example 4.3.2 is an $\omega$-homeomorphism but not a gc-homeomorphism, because $f^{-1}$ is not a gc irresolute map.

**Remark 4.3.15** We obtain the following implications from the above discussions and known results:

\[
\begin{array}{c}
\text{gc-homeomorphism} \\
\Rightarrow \\
\omega^*\text{-homeomorphism} \\
\Rightarrow \\
\omega\text{-homeomorphism} \\
\Rightarrow \\
g\text{-homeomorphism}
\end{array}
\]

**Proposition 4.3.16** If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are $\omega^*$-homeomorphisms, then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is also $\omega^*$-homeomorphism.

**Proof:** Let $U$ be an $\omega$-open set in $(Z, \eta)$. Now, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, $V$ is $\omega$-open in $(Y, \sigma)$ and so again by hypothesis, $f^{-1}(V)$ is $\omega$-open in $(X, \tau)$. Therefore, $g \circ f$ is...
Also for an \( \omega \)-open set \( G \) in \((X, \tau)\), we have \((g \circ f)(G) = g(f(G)) = g(W)\), where \( W = f(G) \). By hypothesis \( f(G) \) is \( \omega \)-open in \((Y, \sigma)\) and so again by hypothesis, \( g(f(G)) \) is \( \omega \)-open in \((Z, \eta)\), i.e., \((g \circ f)(G) \) is \( \omega \)-open in \((Z, \eta)\) and therefore \( (g \circ f)^{-1} \) is \( \omega \)-irresolute. Hence \( g \circ f \) is an \( \omega \)-homeomorphism.

**Theorem 4.3.17** The set \( \omega \)-h\((X, \tau)\) is a group under the composition of maps.

**Proof:** Define a binary operation \( \cdot : \omega \)-h\((X, \tau) \times \omega \)-h\((X, \tau) \to \omega \)-h\((X, \tau)\) by \( f \cdot g = g \circ f \) for all \( f, g \in \omega \)-h\((X, \tau)\) and \( \cdot \) is the usual operation of composition of maps. Then by Proposition 4.3.16, \( g \circ f \in \omega \)-h\((X, \tau)\). We know that the composition of maps is associative and the identity map \( I : (X, \tau) \to (X, \tau) \) belonging to \( \omega \)-h\((X, \tau)\) serves as the identity element. If \( f \in \omega \)-h\((X, \tau)\), then \( f^{-1} \in \omega \)-h\((X, \tau)\) such that \( f \circ f^{-1} = f^{-1} \circ f = I \) and so inverse exists for each element of \( \omega \)-h\((X, \tau)\). Therefore, \((\omega \)-h\((X, \tau)\), \( \cdot \)\) is a group under the operation of composition of maps.

**Theorem 4.3.18** Let \( f : (X, \tau) \to (Y, \sigma) \) be an \( \omega \)-homeomorphism. Then \( f \) induces an isomorphism from the group \( \omega \)-h\((X, \tau)\) onto the group \( \omega \)-h\((Y, \sigma)\).

**Proof:** Using the map \( f \), we define a map \( \psi_f : \omega \)-h\((X, \tau) \to \omega \)-(Y,\( \sigma)\) by \( \psi_f(h) = f \circ h \circ f^{-1} \) for every \( h \in \omega \)-h\((X, \tau)\). Then \( \psi_f \) is a bijection. Further, for all \( h_1, h_2 \in \omega \)-h\((X, \tau)\), \( \psi_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \psi_f(h_1) \circ \psi_f(h_2) \). Therefore, \( \psi_f \) is a homomorphism and so it is an isomorphism induced by \( f \).

**Theorem 4.3.19** \( \omega \)-homeomorphism is an equivalence relation in the collection of all topological spaces.

**Proof:** Reflexivity and symmetry are immediate and transitivity follows from Proposition 4.3.16.
**Theorem 4.3.20** If \( f : (X, \tau) \to (Y, \sigma) \) is an \( \omega \)-homeomorphism, then 
\[
\omega \text{-cl}(f^{-1}(B)) = f^{-1}(\omega \text{-cl}(B))
\]
for all \( B \subseteq Y \).

**Proof:** Since \( f \) is an \( \omega \)-homeomorphism, \( f \) is \( \omega \)-irresolute. Since \( \omega \text{-cl}(f(B)) \) is an \( \omega \)-closed set in \((Y, \sigma)\), \( f^{-1}(\omega \text{-cl}(f(B))) \) is \( \omega \)-closed in \((X, \tau)\). Now, 
\[
f^{-1}(B) \subseteq f^{-1}(\omega \text{-cl}(B))
\]
and so by Proposition 2.3.9, \( \omega \text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\omega \text{-cl}(B)) \).

Again since \( f \) is an \( \omega \)-homeomorphism, \( f^{-1} \) is \( \omega \)-irresolute. Since 
\[
\omega \text{-cl}(f^{-1}(B)) \text{ is } \omega \text{-closed in } (X, \tau),
\]
\[
(f^{-1})^{-1}(\omega \text{-cl}(f^{-1}(B))) = f(\omega \text{-cl}(f^{-1}(B)))
\]
is \( \omega \)-closed in \((Y, \sigma)\). Now, \( B \subseteq (f^{-1})^{-1}(f^{-1}(B)) \subseteq (f^{-1})^{-1}(\omega \text{-cl}(f^{-1}(B))) = f(\omega \text{-cl}(f^{-1}(B))) \) and so \( \omega \text{-cl}(B) \subseteq f(\omega \text{-cl}(f^{-1}(B))) \). Therefore, 
\[
f^{-1}(\omega \text{-cl}(B)) \subseteq f^{-1}(f(\omega \text{-cl}(f^{-1}(B)))) \subseteq \omega \text{-cl}(f^{-1}(B))
\]
and hence the equality holds.

**Corollary 4.3.21** If \( f : (X, \tau) \to (Y, \sigma) \) is an \( \omega \)-homeomorphism, then 
\[
\omega \text{-cl}(f(B)) = f(\omega \text{-cl}(B))
\]
for all \( B \subseteq X \).

**Proof:** Since \( f : (X, \tau) \to (Y, \sigma) \) is an \( \omega \)-homeomorphism, \( f^{-1} : (Y, \sigma) \to (X, \tau) \) is also an \( \omega \)-homeomorphism. Therefore by Theorem 4.3.20, 
\[
\omega \text{-cl}(f^{-1}(B)) = (f^{-1})^{-1}(\omega \text{-cl}(B))
\]
for all \( B \subseteq X \). i.e., \( \omega \text{-cl}(f(B)) = f(\omega \text{-cl}(B)) \).

**Corollary 4.3.22** If \( f : (X, \tau) \to (Y, \sigma) \) is an \( \omega \)-homeomorphism, then 
\[
f(\omega \text{-int}(B)) = \omega \text{-int}(f(B))
\]
for all \( B \subseteq X \).

**Proof:** By Proposition 2.4.18, for any set \( B \subseteq X \), \( \omega \text{-int}(B) = (\omega \text{-cl}(B^c))^c \).

Thus 
\[
f(\omega \text{-int}(B)) = f((\omega \text{-cl}(B^c))^c)
\]
\[
= (f(\omega \text{-cl}(B^c)))^c
\]
\[
= (\omega \text{-cl}(f(B^c)))^c, \text{ by Corollary 4.3.21.}
\]
\[
= (\omega \text{-cl}((f(B))^c))^c = \omega \text{-int}(f(B)) \text{ by Proposition 2.4.18.}
\]

**Corollary 4.3.23** If \( f : (X, \tau) \to (Y, \sigma) \) is an \( \omega \)-homeomorphism, then 
\[
f^{-1}(\omega \text{-int}(B)) = \omega \text{-int}(f^{-1}(B))
\]
for all \( B \subseteq Y \).

**Proof:** Since \( f^{-1} : (Y, \sigma) \to (X, \tau) \) is also an \( \omega \)-homeomorphism, the proof follows from Corollary 4.3.22.
4.4. ω-Compactness and ω-connectedness

Di Maio and Noiri [26] used semi-open covers to introduce a new class of compact spaces called s-closed spaces. Sundaram [112] introduced the concepts of GO-compact spaces and GO-connected spaces by using g-open sets in topological spaces. Recently, Devi [21] introduced the notions of GαO-compactness, GαO-connectedness, αGO-compactness and αGO-connectedness using gα-open and αg-open sets. In this section, we introduce ω-compactness and ω-connectedness using ω-open sets and study some of their properties.

**Definition 4.4.1** A collection \( \{A_i : i \in \Lambda\} \) of ω-open sets in a topological space \((X, \tau)\) is called an ω-open cover of a subset \(A\) in \((X, \tau)\) if \(A \subseteq \bigcup_{i \in \Lambda} A_i\).

**Definition 4.4.2** A topological space \((X, \tau)\) is called ω-compact if every ω-open cover of \((X, \tau)\) has a finite ω-subcover.

**Definition 4.4.3** A subset \(A\) of a topological space \((X, \tau)\) is called ω-compact relative to \((X, \tau)\), if for every collection \(\{A_i : i \in \Lambda\}\) of ω-open subsets of \((X, \tau)\) such that \(A \subseteq \bigcup_{i \in \Lambda} A_i\), there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(A \subseteq \bigcup_{i \in \Lambda_0} A_i\).

**Definition 4.4.4** A subset \(A\) of a topological space \((X, \tau)\) is called ω-compact if \(A\) is ω-compact as a subspace of \((X, \tau)\).

**Proposition 4.4.5** An ω-closed subset of an ω-compact space is ω-compact relative to \((X, \tau)\).

**Proof**: Let \(A\) be an ω-closed subset of an ω-compact space \((X, \tau)\). Then \(A^c\) is ω-open in \((X, \tau)\). Let \(C\) be an ω-open cover of \(A\) in \((X, \tau)\). Therefore \(C\) along with \(A^c\) form an ω-open cover of \((X, \tau)\). Since \((X, \tau)\) is ω-compact, it has a finite sub cover, say \(\{V_1, V_2, \ldots, V_n\}\). If this subcover contains \(A^c\),
we discard it. Otherwise leave the subcover as it is. Thus we have obtained a finite subcover of A and so A is $\omega$-compact relative to $(X, \tau)$.

**Proposition 4.4.6** A $\omega$-closed subset of GO-compact space is GO-compact relative to $(X, \tau)$.

**Proof:** By Proposition 2.2.4, every $\omega$-closed set is $g$-closed and since a g-closed subset of a GO-compact space is GO-compact relative to $(X, \tau)$ ([112], Theorem 5.5) the result follows.

**Proposition 4.4.7** An $\omega$-continuous image of an $\omega$-compact space is compact.

**Proof:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an $\omega$-continuous onto map, where $(X, \tau)$ is an $\omega$-compact space. Let $\{A_i : i \in \Lambda\}$ be an open cover of $(Y, \sigma)$. Then $\{f^{-1}(A_i) : i \in \Lambda\}$ is an $\omega$-open cover of $(X, \tau)$. Since $(X, \tau)$ is $\omega$-compact, it has a finite subcover, say $\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$. Since f is onto, $\{A_1, A_2, \ldots, A_n\}$ is an open cover of $(Y, \sigma)$ and so $(Y, \sigma)$ is compact.

**Proposition 4.4.8** If a map $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\omega$-irresolute and a subset $B$ is $\omega$-compact relative to $(X, \tau)$, then the image $f(B)$ is $\omega$-compact relative to $(Y, \sigma)$.

**Proof:** Let $\{A_i : i \in \Lambda\}$ be any collection of $\omega$-open sets of $(Y, \sigma)$ such that $f(B) \subseteq \bigcup_{i \in \Lambda} A_i$. Then $B \subseteq \bigcup_{i \in \Lambda} f^{-1}(A_i)$. By hypothesis, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $B \subseteq \bigcup_{i \in \Lambda_0} f^{-1}(A_i)$. Therefore $f(B) \subseteq \bigcup_{i \in \Lambda_0} A_i$ and so $f(B)$ is $\omega$-compact relative to $(Y, \sigma)$.

**Proposition 4.4.9** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a strongly $\omega$-continuous onto map where $(X, \tau)$ is a compact space, then $(Y, \sigma)$ is $\omega$-compact.

**Proof:** Let $\{A_i : i \in \Lambda\}$ be an $\omega$-open cover of $(Y, \sigma)$. Then $\{f^{-1}(A_i) : i \in \Lambda\}$ is an open cover of $(X, \tau)$, since $f$ is strongly $\omega$-continuous. Since $(X, \tau)$ is compact, it has a finite subcover say, $\{f^{-1}(A_1), f^{-1}(A_2), \ldots\}$.
We introduce \( \omega \)-connected spaces in topological spaces and study some of their properties.

**Definition 4.4.13** A topological space \((X, \tau)\) is called an \( \omega \)-connected space if \((X, \tau)\) cannot be written as a disjoint union of two non-empty \( \omega \)-open sets.

A subset of \((X, \tau)\) is \( \omega \)-connected if it is \( \omega \)-connected as subspace of \((X, \tau)\).

**Theorem 4.4.14** For a topological space \((X, \tau)\), the following are equivalent:

i) \((X, \tau)\) is \( \omega \)-connected.

ii) The only subsets of \((X, \tau)\) which are both \( \omega \)-open and \( \omega \)-closed are the empty set \( \emptyset \) and \( X \).
iii) Each ω-continuous map of \((X, \tau)\) into a discrete space \((Y, \sigma)\) with at least two points is a constant map.

**Proof:** i) ⇒ ii) : Let \(U\) be an ω-open and ω-closed subset of \((X, \tau)\). Then \(U^c\) is both ω-open and ω-closed in \((X, \tau)\). Since \((X, \tau)\) is the disjoint union of the ω-open sets \(U\) and \(U^c\), by assumption one of these must be empty, i.e., \(U = \emptyset\) or \(U = X\).

ii) ⇒ i) : Suppose that \(X = A \cup B\) where \(A\) and \(B\) are disjoint non-empty ω-open subsets of \((X, \tau)\). Then \(A\) is both ω-open and ω-closed subset of \((X, \tau)\) and therefore by assumption, \(A = \emptyset\) or \(X\). Thus \((X, \tau)\) is ω-connected.

ii) ⇒ iii) : Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be an ω-continuous map. Then \((X, \tau)\) is covered by ω-open and ω-closed covering \(\{f^{-1}(y) : y \in Y\}\). By assumption \(f^{-1}(y) = \emptyset\) or \(X\) for each \(y \in Y\). If \(f^{-1}(y) = \emptyset\) for each \(y \in Y\), then \(f\) fails to be a map. Therefore there exists at least one point say, \(y_1 \in Y\) such that \(f^{-1}(y_1) \neq \emptyset\) and hence \(f^{-1}(y_1) = X\), which shows that \(f\) is a constant map.

iii) ⇒ ii) : Let \(U\) be both ω-open and ω-closed in \((X, \tau)\). Suppose that \(U \neq \emptyset\). Define \(f : (X, \tau) \rightarrow (Y, \sigma)\) by \(f(U) = \{y_1\}\) and \(f(U^c) = \{y_2\}\) for some distinct points \(y_1\) and \(y_2\) in \((Y, \sigma)\), then \(f\) is an ω-continuous map. Therefore, by assumption \(f\) is a constant map. Therefore \(y_1 = y_2\) and so \(U = X\).

**Proposition 4.4.15** Every ω-connected space is connected but not conversely.

**Proof:** Let \((X, \tau)\) be an ω-connected space. Suppose that \((X, \tau)\) is not connected. Then \(X = A \cup B\) where \(A\) and \(B\) are disjoint nonempty open sets in \((X, \tau)\). By Proposition 2.4.2, \(A\) and \(B\) are ω-open and \(X = A \cup B\), where \(A\) and \(B\) are disjoint nonempty and ω-open sets in \((X, \tau)\). This contradicts the fact that \((X, \tau)\) is ω-connected and so \((X, \tau)\) is connected.

**Example 4.4.16** Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, X\}\). Then \((X, \tau)\) is a connected space but not an ω-connected space, because \(X = \{a\} \cup \{b, c\}\),
where \{a\} and \{b, c\} are \(\omega\)-open sets in \((X, \tau)\).

**Proposition 4.4.17** If \((X, \tau)\) is a \(T_\omega\)-space and connected, then \((X, \tau)\) is 
\(\omega\)-connected.

The proof is easy consequences of definitions.

**Proposition 4.4.18** If \(f: (X, \tau) \rightarrow (Y, \sigma)\) is an \(\omega\)-continuous surjection and 
\((X, \tau)\) is \(\omega\)-connected, then \((Y, \sigma)\) is connected.

**Proof:** Suppose that \(Y = A \cup B\), where \(A\) and \(B\) are disjoint nonempty 
open sets of \((Y, \sigma)\). Since \(f\) is \(\omega\)-continuous and onto, 
\(X = f^{-1}(A) \cup f^{-1}(B)\) where, \(f^{-1}(A)\) and \(f^{-1}(B)\) are disjoint nonempty \(\omega\)-open sets in \((X, \tau)\). This 
contradicts the fact that \((X, \tau)\) is \(\omega\)-connected and so \((Y, \sigma)\) is connected.

**Proposition 4.4.19** If \(f: (X, \tau) \rightarrow (Y, \sigma)\) is an \(\omega\)-irresolute surjection and 
\((X, \tau)\) is \(\omega\)-connected, then \((Y, \sigma)\) is \(\omega\)-connected.

**Proof:** Similar to Proposition 4.4.18.

**Proposition 4.4.20** If \(f: (X, \tau) \rightarrow (Y, \sigma)\) is strongly \(\omega\)-continuous onto 
map, where \((X, \tau)\) is a connected space, then \((Y, \sigma)\) is \(\omega\)-connected.

**Proof:** Similar to Proposition 4.4.18.

**Proposition 4.4.21** If \((X, \tau)\) is a topological space with atleast two points 
and if \(SO(X, \tau) \equiv \{F \subseteq X : F^c \in \tau\}\), then \((X, \tau)\) is not \(\omega\)-connected.

**Proof:** By Theorem 2.2.41, there exists a proper subset of \((X, \tau)\) which is 
both \(\omega\)-open and \(\omega\)-closed. Therefore by Theorem 4.4.14, \((X, \tau)\) is not 
\(\omega\)-connected.

**Proposition 4.4.22** If \(f: (X, \tau) \rightarrow (Y, \sigma)\) is an \(\omega\)-continuous map, then \(f(H)\) 
is a connected subset of \((Y, \sigma)\) for every \(\omega\)-closed and \(\omega\)-connected subset \(H\) 
of \((X, \tau)\).

**Proof:** The restriction \(f_H\) of \(f\) to \(H\) is \(\omega\)-continuous by Proposition 3.2.22.

By Proposition 4.4.18, the image of the \(\omega\)-connected space \((H, \tau_H)\) under 
\(f_H: (H, \tau_H) \rightarrow (f(H), \sigma_{f(H)})\) is connected. Therefore \((f(H), \sigma_{f(H)})\) is connected.
Thus $f(H)$ is a connected subset of $(Y, \sigma)$.

### 4.5. $\omega$-regular and $\omega$-normal spaces

Munshi [78] introduced $g$-regular and $g$-normal spaces using $g$-closed sets in topological spaces. Noiri and Popa [92] have further investigated the results of Munshi. In this section we introduce $\omega$-regular spaces and $\omega$-normal spaces in topological spaces. We obtain several characterizations of $\omega$-regular and $\omega$-normal spaces.

**Definition 4.5.1** A space $(X, \tau)$ is said to be $\omega$-regular if for every $\omega$-closed set $F$ and a point $x \in F$, there exists disjoint open sets $U$ and $V$ such that $F \subseteq U$ and $x \in V$.

**Remark 4.5.2** It is obvious that every $\omega$-regular space is regular but not conversely. Consider the topological space $(X, \tau)$ of Example 4.2.6. Then $(X, \tau)$ is a regular space but not an $\omega$-regular space.

**Proposition 4.5.3** Let $(X, \tau)$ be a topological space. Then the following are equivalent:

i) $(X, \tau)$ is an $\omega$-regular space.

ii) For each $x \in X$ and each $\omega$-open neighbourhood $W$ of $x$ there exists an open neighbourhood $V$ of $x$ such that $\text{cl}(V) \subseteq W$.

**Proof:** i) $\Rightarrow$ ii): Let $W$ be any $\omega$-open neighbourhood of $x$. Then there exists an $\omega$-open set $G$ such that $x \in G \subseteq W$. Since $G^c$ is $\omega$-closed and $x \notin G^c$, by hypothesis there exists open sets $U$ and $V$ such that $G^c \subseteq U$, $x \in V$ and $U \cap V = \emptyset$ and so $V \subseteq U^c$. Now, $\text{cl}(V) \subseteq \text{cl}(U^c) = U^c$ and $G^c \subseteq U$ implies $U^c \subseteq G \subseteq W$. Therefore $\text{cl}(V) \subseteq W$.
ii) $\Rightarrow$ i): Let $F$ be any $\omega$-closed set and $x \notin F$. Then $x \in F^c$ and $F^c$ is $\omega$-open and so $F^c$ is an $\omega$-neighbourhood of $x$. By hypothesis, there exists an open neighbourhood $V$ of $x$ such that $x \in V$ and $\text{cl}(V) \subseteq F^c$, which implies $F \subseteq (\text{cl}(V))^c$. Then $(\text{cl}(V))^c$ is an open set containing $F$ and $V \cap (\text{cl}(V))^c = \emptyset$. Therefore $X$ is $\omega$-regular.

**Proposition 4.5.4** For a space $(X, \tau)$ the following are equivalent:

i) $(X, \tau)$ is normal.

ii) For every pair of disjoint closed sets $A$ and $B$, there exists $\omega$-open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

**Proof:** i) $\Rightarrow$ ii): Let $F$ and $K$ be disjoint closed subsets of $(X, \tau)$. By hypothesis, there exists disjoint open sets (and hence $\omega$-open sets) $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

ii) $\Rightarrow$ i): Let $A$ and $B$ be closed subsets of $(X, \tau)$. Then by assumption, $A \subseteq G$, $B \subseteq H$ and $G \cap H = \emptyset$ where $G$ and $H$ are disjoint $\omega$-open sets. Since $A$ and $B$ are semi closed, by Theorem 2.4.6, $A \subseteq \text{int}(G)$ and $B \subseteq \text{int}(H)$. Further, $\text{int}(G) \cap \text{int}(H) = \text{int}(G \cap H) = \emptyset$.

**Proposition 4.5.5** If $(X, \tau)$ is semi-normal, then the following statements are true:

i) For each semi-closed set $A$ and every $\omega$-open set $B$ such that $A \subseteq B$, there exists a semi-open set $U$ such that $A \subseteq U \subseteq \text{scl}(U) \subseteq B$.

ii) For every $\omega$-closed set $A$ and every semi-open set $B$ containing $A$, there exists a semi-open set $U$ such that $A \subseteq U \subseteq \text{scl}(U) \subseteq B$.

iii) For every pair consisting of disjoint sets $A$ and $B$ one of which is semi-closed and the other is $\omega$-closed there exists semi-open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $\text{scl}(U) \cap \text{scl}(V) = \emptyset$.

**Proof:** i). $A$ be a semi-closed set and $B$ be an $\omega$-open set such that $A \subseteq B$. Then $A \cap B^c = \emptyset$, where $A$ is semi-closed and $B^c$ is $\omega$-closed. Therefore,
by Proposition 2.2.46, there exists semi-open sets $U$ and $V$ such that $A \subseteq U$, $B^c \subseteq V$ and $U \cap V = \emptyset$. Thus $A \subseteq U \subseteq V^c \subseteq B$. Since $V^c$ is semi-closed, $scl(U) \subseteq V^c$ and so $A \subseteq U \subseteq scl(U) \subseteq B$.

ii). Let $A$ be an $\omega$-closed set and $B$ be a semi-open set such that $A \subseteq B$. Then $B^c \subseteq A^c$. Since $(X, \tau)$ is semi-normal and $A^c$ is an $\omega$-open set containing the semi-closed set $B^c$, we have by i), there exists a semi-open set $G$ such that $B^c \subseteq G \subseteq scl(G) \subseteq A^c$. Thus $A \subseteq (scl(G))^c \subseteq G^c \subseteq B$. Let $U = (scl(G))^c$. Then $U$ is semi-open and $A \subseteq U \subseteq scl(U) \subseteq B$.

iii). Let $A$ be an $\omega$-closed set and $B$ be a semi-closed set such that $A \cap B = \emptyset$. Then $A \subseteq B^c$ and $B^c$ is semi-open. Since $(X, \tau)$ is semi-normal, we have by ii), there exists a semi-open set $S$ such that $A \subseteq S \subseteq scl(S) \subseteq B^c$. Since $A$ is $\omega$-closed and $S$ is semi-open, we have again by ii), there is a semi-open set $U$ such that $A \subseteq U \subseteq scl(U) \subseteq S$. Thus $A \subseteq U \subseteq scl(U) \subseteq S \subseteq scl(S) \subseteq B^c$. Let $V = (scl(S))^c$. Thus $V$ is semi-open, $B \subseteq V$ and $scl(U) \cap scl(V) = \emptyset$.

**Theorem 4.5.6[78]** A space $(X, \tau)$ is symmetric if and only if $\{x\}$ is $g$-closed in $(X, \tau)$ for each point $x$ of $(X, \tau)$.

**Theorem 4.5.7** A $gT_\omega$ space $(X, \tau)$ is symmetric if and only if $\{x\}$ is $\omega$-closed in $(X, \tau)$ for each point $x$ of $(X, \tau)$.

**Proof**: Follows from Theorem 4.5.6.

**Proposition 4.5.8** Every semi-normal, symmetric and $gT_\omega$ space $(X, \tau)$ is $s$-regular.

**Proof**: Let $F$ be a closed subset of $(X, \tau)$ and $x \in X$ such that $x \notin F$. Since $(X, \tau)$ is symmetric and $gT_\omega$, by Theorem 4.5.7, $\{x\}$ is $\omega$-closed. Since $F$ is closed, it is semi-closed and since $(X, \tau)$ is semi-normal, we have by Proposition 2.2.46, there exists disjoint semi-open sets $U_1$ and $U_2$ such that $F \subseteq U_1$ and $\{x\} \subseteq U_2$. Therefore $(X, \tau)$ is $s$-regular.
**Proposition 4.5.9** If $(X, \tau)$ is an $\omega$-regular space and $Y$ is an open and $\omega$-closed subset of $(X, \tau)$, then the subspace $Y$ is $\omega$-regular.

**Proof**: Let $F$ be any $\omega$-closed subset of $Y$ and $y \in F^c$. By Proposition 2.2.35, $F$ is $\omega$-closed in $(X, \tau)$. Since $(X, \tau)$ is $\omega$-regular there exist disjoint open sets $U$ and $V$ of $(X, \tau)$ such that $y \in U$ and $F \subseteq V$. Therefore $U \cap Y$ and $V \cap Y$ are disjoint open sets of the subspace $Y$ such that $y \in U \cap Y$ and $F \subseteq V \cap Y$. Hence the subspace $Y$ is $\omega$-regular.

**Theorem 4.5.10** A topological space $(X, \tau)$ is $\omega$-regular if and only if for each $\omega$-closed set $F$ of $(X, \tau)$ and each $x \in F^c$ there exist open sets $U$ and $V$ of $(X, \tau)$ such that $x \in U$, $F \subseteq V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

**Proof**: Let $F$ be an $\omega$-closed set of $(X, \tau)$ and $x \not\in F$. Then there exist open sets $U_0$ and $V$ of $(X, \tau)$ such that $x \in U_0$, $F \subseteq V$ and $U_0 \cap V = \emptyset$, which implies $U_0 \cap \text{cl}(V) = \emptyset$. Since $\text{cl}(V)$ is closed, it is $\omega$-closed and $x \not\in \text{cl}(V)$. Since $(X, \tau)$ is $\omega$-regular, there exist open sets $G$ and $H$ of $(X, \tau)$ such that $x \in G$, $\text{cl}(V) \subseteq H$ and $G \cap H = \emptyset$, which implies $\text{cl}(G) \cap H = \emptyset$. Let $U = U_0 \cap G$, then $U$ and $V$ are open sets of $(X, \tau)$ such that $x \in U$, $F \subseteq V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

Converse part is trivial.

**Corollary 4.5.11** If a space $(X, \tau)$ is $\omega$-regular, symmetric and $\mathcal{T}_0$, then it is Urysohn.

**Proof**: Let $x$ and $y$ be any two distinct points of $(X, \tau)$. Since $(X, \tau)$ is symmetric and $\mathcal{T}_0$, $\{x\}$ is $\omega$-closed by Theorem 4.5.7. Therefore, by Theorem 4.5.10, there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

**Corollary 4.5.12** If a space $(X, \tau)$ is $\mathcal{T}_0$, $\omega$-regular and symmetric then it is Hausdorff.

**Proof**: Similar to Corollary 4.5.11.
**Theorem 4.5.13** Let \((X, \tau)\) be a topological space. Then the following statements are equivalent:

i) \((X, \tau)\) is \(\omega\)-regular.

ii) For each point \(x \in X\) and for each \(\omega\)-open neighbourhood \(W\) of \(x\), there exists an open neighbourhood \(U\) of \(x\) such that \(\text{cl}(U) \subseteq W\).

iii) For each point \(x \in X\) and for each \(\omega\)-closed set \(F\) not containing \(x\), there exists an open neighbourhood \(V\) of \(x\) such that \(\text{cl}(V) \cap F = \emptyset\).

**Proof:** By Proposition 4.5.3, i) \(\Leftrightarrow\) ii).

ii) \(\Rightarrow\) iii): Let \(x \in X\) and \(F\) be an \(\omega\)-closed set such that \(x \notin F\). Then \(F^c\) is an \(\omega\)-open neighbourhood of \(x\) and by hypothesis, there exists an open neighbourhood \(V\) of \(x\) such that \(\text{cl}(V) \subseteq F^c\) and hence \(\text{cl}(V) \cap F = \emptyset\).

iii) \(\Rightarrow\) ii): Let \(x \in X\) and \(W\) be an \(\omega\)-open neighbourhood of \(x\). Then there exists an \(\omega\)-open set \(G\) such that \(x \in G \subseteq W\). Since \(G^c\) is \(\omega\)-closed and \(x \notin G^c\), by hypothesis there exists an open neighbourhood \(U\) of \(x\) such that \(\text{cl}(U) \cap G^c = \emptyset\). Therefore \(\text{cl}(U) \subseteq G \subseteq W\).

**Theorem 4.5.14** The following are equivalent for a space \((X, \tau)\).

i) \((X, \tau)\) is \(\omega\)-regular,

ii) \(\text{cl}_\omega(A) = \omega\text{-cl}(A)\) for each subset \(A\) of \((X, \tau)\),

iii) \(\text{cl}_\omega(A) = A\) for each \(\omega\)-closed set \(A\).

**Proof:** i) \(\Rightarrow\) ii): For any subset \(A\) of \((X, \tau)\), we have always \(A \subseteq \omega\text{-cl}(A) \subseteq \text{cl}_\omega(A)\). Let \(x \in (\omega\text{-cl}(A))^c\). Then there exists an \(\omega\)-closed set \(F\) such that \(x \in F^c\) and \(A \subseteq F\). By assumption, there exists disjoint open sets \(U\) and \(V\) such that \(x \in U\) and \(F \subseteq V\). Now, \(x \in U \subseteq \text{cl}(U) \subseteq V^c \subseteq F^c \subseteq A^c\) and therefore \(\text{cl}(U) \cap A = \emptyset\). Thus \(x \in (\text{cl}_\omega(A))^c\) and hence \(\text{cl}_\omega(A) = \omega\text{-cl}(A)\).

ii) \(\Rightarrow\) iii): is trivial.

iii) \(\Rightarrow\) i): Let \(F\) be any \(\omega\)-closed set and \(x \in F^c\). Since \(F\) is \(\omega\)-closed, by
assumption $x \in (\text{cl}_\theta(F))^c$ and so there exists an open set $U$ such that $x \in U$ and $\text{cl}(U) \cap F = \emptyset$. Then $F \subseteq (\text{cl}(U))^c$. Let $V = (\text{cl}(U))^c$. Then $V$ is an open such that $F \subseteq V$. Also the sets $U$ and $V$ are disjoint and hence $(X, \tau)$ is $\omega$-regular.

**Proposition 4.5.15** If $(X, \tau)$ is an $\omega$-regular space and $f : (X, \tau) \to (Y, \sigma)$ is bijective, pre-semi-open, $\omega$-continuous and open then $(Y, \sigma)$ is $\omega$-regular.

**Proof:** Let $F$ be any $\omega$-closed subset of $(Y, \sigma)$ and $y \notin F$. Since the map $f$ is $\omega$-irresolute by Proposition 3.3.10, we have $f^{-1}(F)$ is $\omega$-closed in $(X, \tau)$. Since $f$ is bijective, let $f(x) = y$, then $x \notin f^{-1}(F)$. By hypothesis, there exists disjoint open sets $U$ and $V$ such that $x \in U$ and $f^{-1}(F) \subseteq V$. Since $f$ is open and bijective, we have $y \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. This shows that the space $(Y, \sigma)$ is also $\omega$-regular.

**Proposition 4.5.16** If $f : (X, \tau) \to (Y, \sigma)$ is irresolute $\omega$-closed continuous injection and $(Y, \sigma)$ is $\omega$-regular, then $(X, \tau)$ is $\omega$-regular.

**Proof:** Let $F$ be any $\omega$-closed set of $(X, \tau)$ and $x \notin F$. Since $f$ is irresolute $\omega$-closed, by Proposition 4.2.8, $f(F)$ is $\omega$-closed in $(Y, \sigma)$ and $f(x) \notin f(F)$. Since $(Y, \sigma)$ is $\omega$-regular and so there exist disjoint open sets $U$ and $V$ in $(Y, \sigma)$ such that $f(x) \in U$ and $f(F) \subseteq V$. i.e., $x \in f^{-1}(U)$, $F \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Therefore $(X, \tau)$ is $\omega$-regular.

**Proposition 4.5.17** If $f : (X, \tau) \to (Y, \sigma)$ is weakly continuous $\omega$-closed injection and $(Y, \sigma)$ is $\omega$-regular, then $(X, \tau)$ is regular.

**Proof:** Let $F$ be any closed set of $(X, \tau)$ and $x \notin F$. Since $f$ is $\omega$-closed, $f(F)$ is $\omega$-closed in $(Y, \sigma)$ and $f(x) \notin f(F)$. Since $(Y, \sigma)$ is $\omega$-regular by Theorem 4.5.10, there exists open sets $U$ and $V$ such that $f(x) \in U$, $f(F) \subseteq V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Since $f$ is weakly continuous it follows that [57, Theorem 1], $x \in f^{-1}(U) \subseteq \text{int}(f^{-1}(\text{cl}(U)))$, $F \subseteq f^{-1}(V) \subseteq \text{int}(f^{-1}(\text{cl}(V)))$ and $\text{int}(f^{-1}(\text{cl}(U))) \cap \text{int}(f^{-1}(\text{cl}(V))) = \emptyset$. Therefore $(X, \tau)$ is regular.
We conclude this section with the introduction of \( \omega \)-normal spaces in topological spaces.

**Definition 4.5.18** A topological space \((X, \tau)\) is said to be \( \omega \)-normal if for any pair of disjoint \( \omega \)-closed sets \( A \) and \( B \), there exist disjoint open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \).

Clearly every \( \omega \)-normal space is normal but not conversely. The space \((X, \tau)\) of Example 4.2.6 is normal but not \( \omega \)-normal.

**Proposition 4.5.19** If \((X, \tau)\) is an \( \omega \)-normal space and \( Y \) is an open and \( \omega \)-closed subset of \((X, \tau)\), then the subspace \( Y \) is \( \omega \)-normal.

**Proof:** Let \( A \) and \( B \) be any disjoint \( \omega \)-closed sets of \( Y \). By Proposition 2.2.35, \( A \) and \( B \) are \( \omega \)-closed in \((X, \tau)\). Since \((X, \tau)\) is \( \omega \)-normal, there exist disjoint open sets \( U \) and \( V \) of \((X, \tau)\) such that \( A \subseteq U \) and \( B \subseteq V \). Then \( A \subseteq U \cap Y \) and \( B \subseteq V \cap Y \) and so the subspace \( Y \) is normal.

In the next theorem we characterize \( \omega \)-normal space.

**Theorem 4.5.20** Let \((X, \tau)\) be a topological space. Then the following statements are equivalent:

i) \((X, \tau)\) is \( \omega \)-normal.

ii) For each \( \omega \)-closed set \( F \) and for each \( \omega \)-open set \( U \) containing \( F \), there exist an open set \( V \) containing \( F \) such that \( \text{cl}(V) \subseteq U \).

iii) For each pair of disjoint \( \omega \)-closed sets \( A \) and \( B \) in \((X, \tau)\), there exists an open set \( U \) containing \( A \) such that \( \text{cl}(U) \cap B = \emptyset \).

iv) For each pair of disjoint \( \omega \)-closed sets \( A \) and \( B \) in \((X, \tau)\), there exists open sets \( U \) containing \( A \) and \( V \) containing \( B \) such that \( \text{cl}(U) \cap \text{cl}(V) = \emptyset \).

**Proof:** i) \( \Rightarrow \) ii): Let \( F \) be an \( \omega \)-closed set and \( U \) be an \( \omega \)-open set such that \( F \subseteq U \). Then \( F \cap U^c = \emptyset \). By assumption, there exist open sets \( V \) and \( W \)...
such that $F \subseteq V$, $U^c \subseteq W$ and $V \cap W = \phi$, which implies $\text{cl}(V) \cap W = \phi$. Now, $\text{cl}(V) \cap U^c \subseteq \text{cl}(V) \cap W = \phi$ and so $\text{cl}(V) \subseteq U$.

ii) $\Rightarrow$ iii): Let $A$ and $B$ be disjoint $\omega$-closed sets of $(X, \tau)$. Since $A \cap B = \phi$, $A \subseteq B^c$ and $B^c$ is $\omega$-open. By assumption, there exists an open set $U$ containing $A$ such that $\text{cl}(U) \subseteq B^c$ and so $\text{cl}(U) \cap B = \phi$.

iii) $\Rightarrow$ iv): Let $A$ and $B$ be any two disjoint $\omega$-closed sets of $(X, \tau)$. Then by assumption, there exists an open set $U$ containing $A$ such that $\text{cl}(U) \cap B = \phi$. Since $\text{cl}(U)$ is closed, it is $\omega$-closed and so $B$ and $\text{cl}(U)$ are disjoint $\omega$-closed sets in $(X, \tau)$. Therefore again by assumption, there exists an open set $V$ containing $B$ such that $\text{cl}(V) \cap \text{cl}(U) = \phi$.

iv) $\Rightarrow$ i): Let $A$ and $B$ be any two disjoint $\omega$-closed sets of $(X, \tau)$. By assumption, there exist open sets $U$ containing $A$ and $V$ containing $B$ such that $\text{cl}(U) \cap \text{cl}(V) = \phi$, we have $U \cap V = \phi$ and thus $(X, \tau)$ is $\omega$-normal.

**Proposition 4.5.21** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is bijective, pre-semi-open, $\omega$-continuous and open and $(X, \tau)$ is $\omega$-normal, then $(Y, \sigma)$ is $\omega$-normal.

**Proof**: Let $A$ and $B$ be any disjoint $\omega$-closed sets of $(Y, \sigma)$. The map $f$ is $\omega$-irresolute by Proposition 3.3.10 and so $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\omega$-closed sets of $(X, \tau)$. Since $(X, \tau)$ is $\omega$-normal, there exists disjoint open sets $U$ and $V$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since $f$ is open and bijective, we have $f(U)$ and $f(V)$ are open in $(Y, \sigma)$ such that $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Therefore $(Y, \sigma)$ is $\omega$-normal.

**Proposition 4.5.22** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is irresolute $\omega$-closed continuous injection and $(Y, \sigma)$ is $\omega$-normal, then $(X, \tau)$ is $\omega$-normal.

**Proof**: Let $A$ and $B$ be any disjoint $\omega$-closed subsets of $(X, \tau)$. Since $f$ is irresolute $\omega$-closed, $f(A)$ and $f(B)$ are disjoint $\omega$-closed sets of $(Y, \sigma)$ by Proposition 4.2.8. Since $(Y, \sigma)$ is $\omega$-normal, there exist disjoint open sets $U$ and $V$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. i.e., $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and
f^{-1}(U) \cap f^{-1}(V) = \emptyset. Since f is continuous f^{-1}(U) and f^{-1}(V) are open in (X, \tau), we have (X, \tau) is \omega-normal.

**Proposition 4.5.23** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is weakly continuous \( \omega \)-closed injection and \((Y, \sigma)\) is \( \omega \)-normal, then \((X, \tau)\) is normal.

**Proof**: Let \( A \) and \( B \) be any two disjoint closed sets of \((X, \tau)\). Since \( f \) is injective and \( \omega \)-closed, \( f(A) \) and \( f(B) \) are disjoint \( \omega \)-closed sets of \((Y, \sigma)\). Since \((Y, \sigma)\) is \( \omega \)-normal, by Theorem 4.5.20, there exist open sets \( U \) and \( V \) such that \( f(A) \subseteq U \), \( f(B) \subseteq V \) and \( cl(U) \cap cl(V) = \emptyset \). Since \( f \) is weakly continuous, it follows that \([57,\text{Theorem 1}]\), \( A \subseteq f^{-1}(U) \subseteq int(f^{-1}(cl(U))) \), \( B \subseteq f^{-1}(V) \subseteq int(f^{-1}(cl(V))) \) and \( int(f^{-1}(cl(U))) \cap int(f^{-1}(cl(V))) = \emptyset \). Therefore, \((X, \tau)\) is normal.