CHAPTER 3

SIMULATION OF STRAPDOWN INERTIAL NAVIGATION SYSTEM USING MODELED AND ANALYSED INERTIAL SENSOR DATA

3.1 INERTIAL SENSORS

Inertial sensors comprise of two primary sensor units: accelerometers and gyroscopes. An inertial measurement unit (IMU) combines multiple accelerometers and gyros, usually three of each, to produce a three-dimensional measurement of specific force and angular rate (Weston and Titterton 2000). New designs of INS employ a strap down architecture, whereby the inertial sensors are fixed with respect to the navigation system casing. The main advantages of the use of strapdown system are the decrease in navigation system size, power and cost. Hence, in this work the strapdown approach is examined.

The sensors used in Strapdown Inertial Navigation System (SDINS) can be generally divided into three groups: navigation, tactical and consumer grade sensors. Low-cost sensors are enabling a new generation of commercial navigation applications especially when aided with other sensors. Among the low-cost sensors, current inertial sensor development is focused on micro-machined electromechanical systems (MEMS) technology. MEMS sensors (Xiaoping 2000) are built using silicon micro-machining techniques which have low part counts and they are relatively cheap to manufacture in large quantities. The advantages of MEMS sensors examined in this work are
smaller in size, low power consumption, less maintenance and price with 
detriment of far less accuracy compared with e.g. optical gyros. The accuracy 
of the MEMS IMU technology is improved with the technique of aided 
navigation.

3.2 MEMS INERTIAL SENSORS

MEMS is probably the most exciting new inertial sensor 
technology. This enables quartz and silicon sensors to be mass produced at 
low cost using etching techniques with several sensors on a single silicon 
wafer. MEMS sensors are small, light, and exhibit much greater shock 
tolerance than the conventional mechanical designs. Apart from size 
reduction, MEMS technology (Gabrielson 1993) offers many benefits such as 
batch production and cost reduction, power (voltage) reduction, robustness, 
and design flexibility, within limits.

However, the reduction in size of the sensing elements creates 
challenges for attaining good performance. In general, as the size decreases, 
the sensitivity (scale factor) decreases, noise increases and driving force 
decreases. Also, the change in Young’s Modulus of silicon is ~100 ppm/°C, 
which leads to thermal sensitivity concerns. However, with the evolution in 
complex computing technology, the errors in the MEMS system can be 
accounted and an inertial unit with accuracy comparable to the optical sensors 
can be achieved using MEMS sensors. As mentioned above, inertial sensors 

3.2.1 MEMS Gyroscopes

Gyroscope may be defined as a system containing a heavy metal 
wheel or rotor, universally mounted so that it has three degrees of freedom. 
This definition holds good for gyroscopes of earlier days. Present day
gyroscopes, like MEMS type, vary in construction but the working principle remains the same. The sensors are used in a variety of roles such as: stabilization, autopilot feedback, flight path sensor or platform stabilization, navigation etc. If a mass is vibrated sinusoidally in a plane which is rotated at some angular rate $\Omega$, then the Coriolis force causes the mass to vibrate sinusoidally perpendicular to the plane with amplitude proportional to $\Omega$. The measurement of the Coriolis-induced motion provides knowledge of $\Omega$. Fundamentally, MEMS gyros fall into four major areas: vibrating beams, vibrating plates, ring resonators and dithered accelerometers. MEMS gyroscopes use the Coriolis Effect (Yazdi et al 1998) by measuring the secondary vibration and calculating angular velocity due to the coriolis force.

3.2.2 MEMS Accelerometers

The acceleration of a vehicle can be determined by measuring the force required to constrain a suspended mass so that it has the same acceleration as the vehicle on which it is suspended. Using Newton’s law, the force ($F$) is equal to mass ($m$) and acceleration ($a$). This acceleration ($a$) is given by the Equation (3.1):

$$F = ma \quad (3.1)$$

By measuring the force required to suspend the mass and knowing its mass, the acceleration can be measured. Whilst it is not practical to determine the acceleration of a vehicle by measuring the total force acting upon it, it is possible to measure the force acting on a small mass contained within the vehicle which is constrained to move with the vehicle. As a result, the mass is displaced with respect to the body.

MEMS accelerometers (Park and Gao 2002) detect acceleration in two primary ways: (i) the displacement of a hinged or flexure-supported proof mass under acceleration results in a change in a capacitive or piezoelectric
readout; (ii) the change in frequency of a vibrating element is caused by a change in the element’s tension induced by a change of loading from a proof mass. The former includes the class known as pendulous accelerometers and the latter are usually known as resonant accelerometers, or VBAs (Vibrating Beam Accelerometers).

### 3.3 INERTIAL SENSOR ERRORS

All types of accelerometers and gyroscopes exhibit biases, scale factor, and cross-coupling errors and random noise to a certain extent. Higher order errors and angular rate acceleration cross-sensitivity may also occur, depending on the sensor type (Nassar and El-Sheimy 1999, El-Diasty et al 2009). Each systematic error source has four components: a fixed contribution, a temperature-dependent variation, a run-to-run variation, and an in-run variation. The fixed contribution is present each time the sensor is used and is corrected by the IMU processor using the laboratory calibration data. The temperature-dependent component can also be corrected by the IMU using laboratory calibration data. The run-to-run variation results in a contribution to the error source that is different each time the sensor is used but remains constant within any run. Finally, the in-run variation contribution to the error source slowly changes during the course of a run. It cannot be corrected by the IMU or by an alignment process. In theory, it can be corrected through integration with other navigation sensors.

The inertial sensor errors can be classified into two parts (Nassar and El-Sheimy 1999), a constant (or deterministic) and a stochastic (or random) part. Major deterministic error sources include bias and scale errors, which can be removed by specific calibration procedures (Park and Gao 2002, Priyanka Aggarwal et al 2006, Aggarwal et al 2008, Zhiqiang and Gebre-Egziabher 2008). However, the inertial sensor random errors primarily include the sensor noise, which consists of two parts, a high frequency and a
low frequency component. The high frequency component has white noise characteristics, while the low frequency component is characterized by correlated noise (Skaloud et al 1999). De-noising methodology is required to filter the high frequency noise in the inertial sensor measurements prior to processing, using a low pass filter, a wavelet or neural de-noising network. Several studies have focused on evaluating such techniques (Nassar and El-Sheimy 2005, Abdel Hamid et al 2004). On the other hand, the low frequency noise component (correlated noise) can be modeled using random processes such as: random constant, random walk, Gauss-Markov (GM) or periodic random processes (Nassar and El-Sheimy 2005).

**Need for inertial sensor error modeling**

The development of the deterministic and stochastic error model for an inertial sensor is one of the most important steps for building a reliable integrated navigation system. The reason is that the inertial sensor propagates large navigation errors in a small time interval. Unless an accurate error model is developed, the mechanization parameters (velocity, attitude, position) will be contaminated by the unmodelled errors and the system performance will be degraded.

**3.4 MEMS INERTIAL SENSOR ERRORS**

As discussed in Section 3.5, the performance characteristics of MEMS inertial sensors (either gyroscopes or accelerometers) are affected by a variety of errors (Zhiqiang Xing and Gebre-Egziabher 2008) as listed below:

a) **Bias**

The bias is the accelerometer /gyro output measured when there is no input acceleration or rotation. The gyro bias is typically expressed in
degree per hour (°/h) or radian per second (rad/s) and the accelerometer bias is expressed in meter per Second Square [m/s² or g].

b) **Scale factor**

Scale factor is the ratio of a change in the input intended to be measured. Scale factor is generally evaluated as the slope of the straight line that can be fit by the method of least squares to input-output data as shown in Figure 3.1 (Titterton and Weston 2005)

![Figure 3.1 Scale Factor and Bias](image)

> **Figure 3.1 Scale Factor and Bias**

c) **Misalignment**

Sensor misalignment errors are a result of mechanical fabrication and manufacturing imperfections in mounting the accelerometer/gyroscope orthogonal triad onto platform.

d) **G-Sensitive drift**

This is a bias which is a function of the current acceleration applied to the sensor.
e) **G²-sensitive drift**

Gyro acceleration-squared sensitive drift rate or gyro anisoelasticity is primarily a function of gyro design and assembly and should not vary much neither from unit to unit nor with time.

f) **CG offset**

An offset is a displacement of the accelerometer signal which can lead to an error in the alignment of the system. The determination of initial roll and pitch angle are affected.

g) **Thermal Sensitivity**

Thermal sensitivity refers to the range of variation of the sensor performance characteristics, particularly bias and scale factor errors, with a change in temperature.

f) **Quantization Noise**

Quantization noise is caused by the small differences between the actual amplitudes of the accelerations and angular rates and the resolution of sensors.

g) **Random errors**

The random errors are basically due to the random variations of the SINS sensor errors (biases) over time. These random processes include white noise, random constant (random bias), random walk and time correlated process.

3.5 **DETERMINISTIC ERROR MODELING**

Error models are often used to analyze the performance of a SDINS from a given set of sensors. Conversely, it may be important to find out how accurate sensors need to be to obtain desired INS performance specifications.
The thermal sensitivity of the MEMS inertial sensors is modeled as follows:

A bias or scale factor correlation with temperature variation can be defined graphically or numerically (using a mathematical expression) through intensive lab thermal testing procedure (El-Diasty et al 2007, Aggarwal et al 2008, Gabrielson 1993). Such correlations can be stored on a computer for online use to provide compensation for temperature variation, provided a thermal sensor is supplied with the sensor (Gulmammadov 2009). The process of characterizing the stochastic variation at different temperatures is one of the most important steps in developing a reliable low cost integrated navigation system (Minha Park 2004, Priyanka Aggarwal et al 2006). Unless an accurate temperature-dependent stochastic model is developed, these errors accumulate with time and degrade the position accuracy if the thermal variations for both accelerometer and gyroscope biases and scale factors are not modeled and compensated. The modified equations (Gulmammadov 2009) for the bias and scale factor variation models are given below in Equations (3.2) and (3.3):

\[
b(t) = b(t_o) + c_1 (t - t_o) + c_2 (t - t_o)^2 + c_3 (t - t_o)^3 \tag{3.2}
\]

\[
S(t) = S(t_o) + d_1 (t - t_o) + d_2 (t - t_o)^2 + d_3 (t - t_o)^3 \tag{3.3}
\]

where, \( t \) corresponds to individual temperature points and \( t_o \) is the room temperature \((25^\circ\text{C})\). \( b(t_o) \) and \( S(t_o) \) are the bias and scale factor values at room temperature, evaluated by 6 position calibration tests (El-Diasty et al 2007), while \( b(t) \) and \( S(t) \) are the evaluated bias and scale factor values for each temperature point where temperature ranges from \(-25^\circ\text{C}\) to \(70^\circ\text{C}\).

The calculated scale factor and biases for accelerometers and gyroscopes at different temperatures are estimated using the Equation (3.2) and (3.3) and are shown in Figure 3.2 and Figure 3.3.
As observed from Figure 3.2, the scale factor variation of accelerometers and gyroscopes with temperature is 0.04 for accelerometers while 3 times for gyroscopes over the whole temperature range.

The constant bias estimated from the conventional six-position static test (as given in the specification of a sensor) was first removed using the calculated bias at different temperature points and then plotted with respect to temperature.
As observed from Figure 3.3, the biases of accelerometers and gyroscopes vary significantly with temperature and hence should be modeled to get accurate navigation results. For accelerometers, the biases drift can be as high as 0.5 m/s\(^2\) while for gyroscopes, the drift in biases can reach 2.5 deg/s over the whole temperature range. Hence there is a need to design an accurate thermal calibration model for low cost MEMS sensors to compensate for these biases and scale factor drift with temperatures. However the improvement in the overall accuracy of the SDINS obtained by using the thermal model depends on the total temperature variation in the raw data.

### 3.5.1 Gyroscope Error Modeling

The error model for all the three gyroscopes used in a strapdown inertial navigation system is represented as shown in Equation (3.4). The other stochastic errors like angle random walk, bias stability, rate random walk, drift rate ramp and time correlated noise are modeled as random noise.

\[
\begin{pmatrix}
\omega_{mi} \\
\omega_{my} \\
\omega_{mz}
\end{pmatrix}
= \begin{pmatrix}
Sb_x & M_{sx} & M_{sz} \\
M_{sx} & Sb_y & M_{sy} \\
M_{sz} & M_{sy} & Sb_z
\end{pmatrix}
\begin{pmatrix}
\omega_n \\
\omega_n \\
\omega_n
\end{pmatrix}
+ \begin{pmatrix}
G_{xx} & G_{xy} & G_{xz} \\
G_{yx} & G_{yy} & G_{yz} \\
G_{zx} & G_{zy} & G_{zz}
\end{pmatrix}
\begin{pmatrix}
a_x \\
a_y \\
a_z
\end{pmatrix}
+ \begin{pmatrix}
G^2_{1x_1 a_x} \\
G^2_{1y_1 a_y} \\
G^2_{1z_1 a_z}
\end{pmatrix}
+ \begin{pmatrix}
b_x \\
b_y \\
b_z
\end{pmatrix}
+ \begin{pmatrix}
Tb_x \\
Tb_y \\
Tb_z
\end{pmatrix}
+ \begin{pmatrix}
\eta_x \\
\eta_y \\
\eta_z
\end{pmatrix}
\]

(3.4)

where, \(\omega_{mi}\) = Measured angular rate along each axis, \(\omega_{ni}\) = True angular rate along each axis, \(a_i\) = Acceleration along each axis, \(Sb_i\) = Scale factor error along each axis, \(M_{ij}\) = Misalignment error along each axis, \(G_{ij}\) = g-sensitive drift between two axes, \(G^2_{1i}\) = \(g^2\)-sensitive drift along each axis, \(b_i\) = Bias error along each axis, \(Tb_i\) = Temperature based drift along each axis (El-Diasty et al 2007), which is given in Equation (3.5), \(\eta_i\) = Random noise along each axis.
\[
\begin{pmatrix}
T_{b_x} \\
T_{b_y} \\
T_{b_z}
\end{pmatrix} = 
\begin{pmatrix}
-0.0049T + 0.2955 \\
-0.0179 + 0.4348 \\
-0.0005T^2 + 0.0425T - 1.0268
\end{pmatrix}
\] (3.5)

### 3.5.2 Accelerometer Error Modeling

The error model for accelerometer triad is represented as in Equation (3.6). The other stochastic errors like velocity random walk, bias stability, drift rate ramp and time correlated noise are modeled as random noise.

\[
\begin{pmatrix}
{a_{mx}} \\
{a_{my}} \\
{a_{mz}}
\end{pmatrix} = 
\begin{pmatrix}
{S_x} & {M_{xy}} & {M_{xz}} \\
{M_{yx}} & {S_y} & {M_{yz}} \\
{M_{zx}} & {M_{zy}} & {S_z}
\end{pmatrix} \begin{pmatrix}
{a_{ix}} \\
{a_{iy}} \\
{a_{iz}}
\end{pmatrix} + 
\begin{pmatrix}
{b_x} \\
{b_y} \\
{b_z}
\end{pmatrix} + 
\begin{pmatrix}
{T_{a_x}} \\
{T_{a_y}} \\
{T_{a_z}}
\end{pmatrix} + 
\begin{pmatrix}
{A_{CGx}} \\
{A_{CGy}} \\
{A_{CGz}}
\end{pmatrix} + 
\begin{pmatrix}
{\eta_x} \\
{\eta_y} \\
{\eta_z}
\end{pmatrix}
\] (3.6)

where, \(a_{mi} = \) Measured acceleration along each axis, \(a_{ui} = \) True acceleration along each axis, \(S_i = \) Scale factor error along each axis, \(M_{ij} = \) Misalignment error along each axis, \(A_{CGi} = CG_i = \) Acceleration due to CG offset along each axis given in Equation (3.7) (El-Diasty et al 2009), \(b_i = \) Bias error along each axis, \(T_{a_i} = \) Temperature based drift along each axis (El-Diasty et al 2007),
given in Equation (3.8)

\[
\begin{pmatrix}
{CG_x} \\
{CG_y} \\
{CG_z}
\end{pmatrix} = 
\begin{pmatrix}
-(q^2 + r^2)X_{Ax} \\
-(p^2 + r^2)Y_{Ay} \\
-(p^2 + q^2)Z_{Az}
\end{pmatrix}
\] (3.7)

\[
\begin{pmatrix}
{T_{a_x}} \\
{T_{a_y}} \\
{T_{a_z}}
\end{pmatrix} = 
\begin{pmatrix}
4.1420 \times 10^{-5}T^3 - 0.0039T^2 + 0.0876T - 0.3803 \\
0.0018T^2 - 0.0236T - 0.5370 \\
1.7881 \times 10^{-4}T^3 - 0.0150T^2 + 0.3564T - 2.2526
\end{pmatrix}
\] (3.8)
3.6 STOCHASTIC MODELING OF INERTIAL SENSOR RANDOM ERRORS

The inertial sensor random errors can be expressed as: white noise, random constant (random bias), random walk, Gauss-Markov (GM) (first and higher orders) random processes (Minha Park 2004). For most of the navigation-grade SDINS systems (gyro drift 0.005-0.01 deg/h), a 1st order Gauss-Markov model is used to describe the random errors associated with inertial sensors (El-Sheimy et al 2004). This is also true for low-cost inertial systems (gyro drift 100-1000 deg/h) although sometimes a white noise process instead of a 1st order GM model is utilized.

3.6.1 Random Processes for Modeling Inertial Sensor Random Errors

In most of the currently used SDINS error models, the inertial sensor random errors are described by a random process (El-Diasty et al 2009). The random errors are basically due to the random variations of the SDINS sensor errors (biases) over time. These random processes include white noise; random constant (random bias); random walk and time correlated process. A white noise process usually has a zero mean and when stationary, it has a constant power spectral density (PSD). In the following equations, the discrete model forms suitable for computer modeling and simulation of time correlated and random walk models are developed by using stochastic differential equation approach as shown in Equation (3.9) to model the random errors of SDINS MEMS sensor errors (El-Diasty et al 2009).

Consider the following vector stochastic differential equation:

\[ dx(t) = A(t)x(t)dt + d\mu(t) \]  \hspace{1cm} (3.9)
where $d\mu(t)$ is a Brownian motion (independent increment) process. In continuous form the above Equation (3.9) is often written as Equation (3.10):

$$\dot{x}(t) = A(t)x(t) + \omega(t) \tag{3.10}$$

This implies that $\omega(t) = \frac{d\mu(t)}{dt}$

Equation (3.10) can also be expressed as given in Equation (3.11):

$$dx(t) = A(t)x(t) + u(t)\sqrt{dt} \tag{3.11}$$

Equation (3.11) will be specialized for two processes, random walk and first order correlated process.

a) **Random walk**

For random walk process $A(t)$ is zero. The random walk can be expressed by the stochastic differential equation given in Equation (3.12):

$$dx_{RW}(t) = u_{RW}(t)\sqrt{dt} \tag{3.12}$$

Else, if implemented in time derivative level in a simulation, then the Equation (3.12) can be modified as given in Equation (3.13)

$$\dot{x}_{RW}(t) = u_{RW}(t) / \sqrt{dt} \tag{3.13}$$

The covariance of this process becomes as given in Equation (3.14)

$$dp_{RW}(t) = q_{RW}(t)dt \tag{3.14}$$

If for instance, this process is the model for the MEMS random drift, $u_{RW}$ is specified in angle error per root time deg/$\sqrt{h}$, then the differential
covariance has units of angle-squared. Monte-Carlo simulation results of implementing Equation (3.14) are presented in Figure 3.4 for several sample paths (time histories) of the random walk process and the mean and standard deviations.

b) **Time correlated process**

Time correlated process is specified by the process time constant $\tau$ and the driving noise $w(t)$ in Equation (3.10). It is desirable to have the variance of stationary process reach a known steady state value by specifying the magnitude of the process noise $Q$ ($P_{ss} = Q$). This is achieved by the following time–invariant scalar stochastic differential equation as given in Equation (3.15).

\[
dx_c = -\frac{1}{\tau} x_c dt + \frac{\sqrt{2}}{\sqrt{\tau}} w_c \sqrt{dt}
\]  
(3.15)

Equation (3.15) is implemented at the time derivative level as given in Equation (3.16)

\[
x_c = -\frac{1}{\tau} x_c + \frac{\sqrt{2}}{\sqrt{\tau}} w_c \sqrt{dt}
\]  
(3.16)

The corresponding variance differential equation is

\[
\dot{p}_c = -\frac{2}{\tau} p_c + \frac{2}{\tau} q_c
\]

The steady state condition $\dot{p}_c = 0$ for this process is $P_{ss} = q_c$

Monte Carlo simulation results implementing Equation (3.16) are presented in Figure 3.5 shown are several sample paths (time histories) of the time correlated process and the corresponding computed standard deviations and mean of the process.
Figure 3.4 Monte Carlo simulation for random walk and its Statistics

Figure 3.5 Monte Carlo simulation for time correlated process and its statistics

From the Figure 3.4 and Figure 3.5 it is clear that the statistical data obtained using the Monte Carlo process for the random walk and time correlated process can be used in the random error modeling of the inertial sensors by adding the noise with the above mean and standard deviation along with the white noise.
3.7 RESULTS AND DISCUSSION

The errors in the MEMS based accelerometers and gyroscopes are modeled by the Equation (3.4) and (3.6) using the various values of the MEMS sensors specification as given in Table (3.1). All the error modeling equations mentioned are added to the sensor data simulated for a trajectory shown in Figure 2.2 and Angle (Velocity) Random Walk, Bias Stability, Rate Random Walk, Drift Rate Ramp, Exponentially Correlated (Markov) Noise are modeled as random noise as discussed in Equation (3.9 – 3.16) and implemented in MATLAB. The results for Roll, Pitch and Yaw rate gyroscopes are shown in Figure (3.6) and X, Y and Z accelerometers are shown in Figure (3.7). Table 3.1 shows the various values of the inertial sensors used in this modeling.

Table 3.1 MEMS gyroscope parameters used in the modeling

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Roll Rate Gyro</th>
<th>Pitch Rate Gyro</th>
<th>Yaw Rate Gyro</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias (deg/hr)</td>
<td>0.1500</td>
<td>0.0900</td>
<td>0.1200</td>
</tr>
<tr>
<td>Scale factor</td>
<td>1.0294</td>
<td>1.0137</td>
<td>0.9838</td>
</tr>
<tr>
<td>Misalignment (deg)</td>
<td>0.0024</td>
<td>0.0157</td>
<td>0.0100</td>
</tr>
<tr>
<td>G sensitive Bias</td>
<td>0.129</td>
<td>0.192</td>
<td>0.125</td>
</tr>
<tr>
<td>G square Sensitive Bias</td>
<td>0.42</td>
<td>0.42</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Table 3.2 MEMS Accelerometer Parameters used in the modeling

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>X axis Accelerometer</th>
<th>Y axis Accelerometer</th>
<th>Z axis Accelerometer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias (g)</td>
<td>1.0294</td>
<td>1.0137</td>
<td>0.9838</td>
</tr>
<tr>
<td>Scale factor</td>
<td>1.0021</td>
<td>1.0008</td>
<td>0.9998</td>
</tr>
<tr>
<td>Misalignment (deg)</td>
<td>0.0188</td>
<td>-0.0146</td>
<td>0.0160</td>
</tr>
</tbody>
</table>
The deterministic and stochastic errors in the accelerometers and gyroscopes are modeled and added to the simulated sensor data to make the simulated data replicate the real sensor data. Thus, the simulated angular rates with errors (for Roll, Pitch and Yaw rate gyroscopes) shown in Figure 3.6 and
simulated accelerations with errors (X, Y and Z accelerometers) shown in Figure 3.7 clearly indicates that the simulated data resembles the real sensor data.

3.8 ERROR ANALYSIS OF INERTIAL SENSORS USING ALLAN VARIANCE

3.8.1 Allan Variance

Allan variance (El-Sheimy et al 2008, Haiying Hou 2004) is a time domain analysis technique used to determine the character of the underlying random processes that give rise to the data noise. Allan variance is a method of representing root mean square (RMS) random drift error as a function of average time and it can provide information on the types and magnitude of various noise terms and the technique is based on the method of cluster analysis (Lawrence 1993). The steps involved in computing the Allan variance is given below:

a) A data stream is divided into clusters of specified length aggregated for specific time period. Assume there are $N$ consecutive data points, each having a sample time of $t_0$. Forming a group of $n$ consecutive data points (with $n < N/2$), each member of the group is a cluster, as shown in Figure (3.8).

![Figure 3.8 Schematic of the Data Structure used in the Derivation of Allan Variance](image)
b) Associated with each other is a time, $T$, which is equal to $n t_0$. If the instantaneous output rate of inertial sensor is $\Omega(t)$ the cluster average can be defined as given in Equation (3.17):

$$\bar{\Omega}_{k}(t) = \frac{1}{T} \int_{t_k}^{t_k + T} \Omega(t)dt$$  \hspace{1cm} (3.17)

where $\bar{\Omega}_{k}(t)$ represents the cluster average of the output rate for a cluster which starts from the $k^{th}$ data point and contains $n$ data points. The definition of the subsequent cluster average is given in Equation (3.18):

$$\bar{\Omega}_{\text{next}}(t) = \frac{1}{T} \int_{t_k}^{t_k + T} \Omega(t)dt$$  \hspace{1cm} (3.18)

where $t_{k+1} = t_k + T$

c) Performing the average operation for each two adjoining cluster and form the differences as given in Equation (3.19)

$$\bar{\xi}_{k\text{=}1\text{=}k} = \bar{\Omega}_{\text{next}}(t) - \bar{\Omega}_{k}(t)$$  \hspace{1cm} (3.19)

For each cluster time $T$, the ensemble of $\bar{\xi}$ defined by the Equation (3.19) forms a set of random variables. The Quantity of interest is the variance of $\bar{\sigma}$ over all the cluster of the same size that can be formed from entire data.

d) Thus, the Allan variance of length $T$ can be defined as (IEEE Std952-1997) given in Equation (3.20):

$$\sigma^2(T) = \frac{1}{2} \left( \bar{\Omega}_{\text{next}}(t) - \bar{\Omega}_{k}(t) \right)^2$$  \hspace{1cm} (3.20)
The brackets in Equation (3.20) denote the averaging operation over the ensemble of clusters. Thus, above equation can be rewritten as given in Equation (3.21):

\[
\sigma^2(T) \frac{1}{2(n - 2n)} \sum_{k=1}^{N-2n} \left[ \overline{\Omega}_{\text{next}}(t) - \overline{\Omega}_k(t) \right]^2
\]

(3.21)

Clearly, for any finite number of data points (N), a finite number of clusters of fixed length (T) can be formed. Hence, Equation (3.21) represents an estimation of quantity \( \sigma^2(T) \) whose quality of estimate depends on the number of independent clusters of fixed length that can be formed.

### 3.8.2 Representation of Noise Terms in Allan Variance

The following subsection will show the integral solution for a number of specific noise terms which are known to exist in the inertial sensor data:

a) **Quantization Noise**

Allan variance for quantization noise is given by Equation (3.22)

\[
\sigma^2(T) = \frac{3Q_z^2}{T^2}
\]

(3.22)

where \( Q_z \) is the quantization noise coefficient \( T \) is the sample interval. Therefore the root Allan variance of the quantization noise when plotted in a log-log scale is represented by slope of -1.

b) **Angle (Velocity) Random Walk**

High frequency noise terms that have correlation time much shorter than the sample time can contribute to the gyro angle or accelerometer
velocity random walk. The Allan variance for angle velocity random walk can be given by Equation (3.23)

\[ \sigma^2(T) = \frac{Q^2}{T} \]  

Equation (3.23) indicates that a log-log plot of \( \sigma^2(T) \) versus \( T \) has a slope of -1/2.

c) **Bias Instability**

The origin of this noise is the electronics or the other components susceptible to random flickering. Because of its low frequency nature it shows as the bias fluctuations in the data. Allan variance for bias instability can be given by Equation (3.24)

\[ \sigma_b^2(\tau) = \frac{2B^2}{\pi} \left[ \ln 2 - \frac{\sin^3(\pi f_0 (\pi f_0 \tau)) + 4\pi f_0 \tau \cos(\pi f_0 \tau)}{2(\pi f_0 \tau)^2} + C_i(2\pi f_0 \tau) - C_i(4\pi f_0 \tau) \right] \]

\[ = \left( \frac{B}{0.6648 f_0} \right)^2 \text{ for } \tau \gg \frac{1}{f_0} \]  

\[(3.24)\]

where \( C_i() \) is the cosine integral function. Thus the bias instability value can be read from the Allan variance plot at the region where the slope is zero.

d) **Sinusoidal Noise**

The Allan variance for sinusoidal noise is given by Equation (3.25)

\[ \sigma_s^2(\tau) = \omega_0^2 \left( \frac{\sin^4(\pi ft)}{(\pi ft)^2} \right)^2 \]  

\[(3.25)\]

Thus the root Allan variance of sinusoid when plotted in log-log scale would indicate sinusoidal behavior with successive peaks attenuated at a slope of -1.
e) Rate Random walk

This noise is result of integrating wideband acceleration PSD. This is a random process of uncertain origin, possibly a limiting case of an exponential correlated noise with a very long correlation time. The Allan variance of rate random walk is given by Equation (3.26):

\[ \sigma_{rrw}^2(f) = \left( x^2 / 3 \right) \tau \]  \hspace{1cm} (3.26)

This indicates that rate of Random Walk is represent by a slope of +1/2 on a log-log plot of \( \sigma(t) \) versus \( \tau \). The unit of K is usually given in deg/hr\(^2\)/(Hz)\(^{1/2}\)

f) Rate Ramp

This is more of a deterministic error than random noise. It could also be due to a very small acceleration of the platform in the same direction and persisting over a long period of time(- hours). Allan Variance of Rate ramp is given by Equation (3.27).

\[ \sigma_{rr}^2(\tau) = \left( \frac{R^2 \tau^2}{2} \right) \]  \hspace{1cm} (3.27)

This indicates that the rate ramp noise has slope of +1 in the log-log plot of \( \sigma(t) \) versus \( \tau \).

3.9 SAMPLE PLOT OF ALLAN VARIANCE

In general, any number of the random process discussed above can be present in the data. Experience shows that in most cases, different noise terms appear in different regions of T and this allows easy identification of
various random processes that exist in the data. With real data, gradual transitions would exist between the different Allan standard deviation slopes. A certain amount of noise or hash would exist in the plot curve of noisy data due to the uncertainty of the measured Allan variance (IEEE Std 1293 1998).

**Representation of Noise Terms in Allan Variance log-log plot**

1. Quantization noise : slope of -1.
3. Bias Instability : slope is zero.
5. Rate Random Walk : slope of +1/2
6. Rate Ramp : slope of +1

### 3.10 RESULTS AND DISCUSSION

Figures (3.9-3.14) show the Allan variance plot for three Gyroscopes and three Accelerometers obtained using the Equations (3.17-3.21).

![Figure 3.9 Allan Variance Plot for Roll rate Gyroscope](image)

Figure 3.9 Allan Variance Plot for Roll rate Gyroscope
Figure 3.10 Allan Variance Plot for Pitch rate Gyroscope

Figure 3.11 Allan Variance Plot for Yaw rate Gyroscope
Figure 3.12 Allan Variance plot for X Accelerometer

Figure 3.13 Allan Variance plot for Y Accelerometer
While examining these figures we can conclude that Quantization noise is the dominant noise because slope is -1 at less correlation time in all of these sensors. When correlation time increases, angle random walk becomes the dominant noise which is represented by a slope of -1/2 in Allan variance plot. As the correlation time further increases the bias instability noise is observed by the zero slope in the Allan variance plot of the sensors. From these Allan variance plots shown in Figures 3.9 – 3.14, the noise coefficients that corrupts the inertial sensor data are identified in the Section (3.12) and this information is useful while modeling the random errors.

3.11 CALCULATION OF NOISE CO-EFFICIENTS FROM ALLAN VARIANCE PLOT

a) Quantization noise calculation

Figures (3.9 – 3.14) clearly indicate that the quantization noise is the dominant noise for short cluster times. Figure 3.15 shows an example of
how to obtain quantization noise coefficient from the Allan variance result in a log-log plot for roll rate gyro.

A lengthy and straightforward calculation (Papoulis 1991) shows the percentage error which is given by Equation (3.28)

$$\sigma(\delta_{AV}) = \frac{1}{\sqrt{2\left(\frac{N}{n} - 1\right)}}$$

(3.28)

where \(N\) is the total number of data points in the entire data set and \(n\) is the number of data points contained in the cluster.

A straight line with slope of \(-1\) fitted to the beginning of the plot meets \(T = 31/2\) hour line at a value of 1.50x10\(^{-7}\) deg which is equal to 5.4x10\(^{-4}\) arc seconds. Since the estimation of quantization noise is based on very short cluster times, the number of independent clusters is very large and the quality of estimation is very good. In fact, even for cluster time as long as \(T = 100\) sec, according to Equation (3.28), the percentage error is only 7.58%.

In fact, the estimation percentage error can be reduced to 2.7%, making the line with slope of \(-1\) to cover only the region from \(T = 0.05\) sec till \(T = 10\) sec. The value of percentage error is equal to 5.4x10\(^{-4}\) x 2.7% = 1.48x10\(^{-5}\) arc seconds.

Thus the quantization coefficient for CIMU Z-axis gyro is estimated as: \(Q = 5.4 \times 10^{-4} \pm 1.48 \times 10^{-5}\) arc seconds.
b) Angle or velocity random walk

In Figures (3.9 – 3.14) there is also a clear indication that the angle random walk is the dominant noise term for long cluster times. There is an example in Figure (3.16) to show how to obtain the random walk coefficients from the Allan variance log-log plot result. A straight line with slope of $-1/2$ is fitted to the long cluster time part of the plot and meets the $T=1$ hour line at a value of $5 \times 10^{-4}$. The unit of angle random walk is deg/h$^{1/2}$. Inspection of the curve shows that the estimation percentage error in this region can reach to 33.36% according to Equation (3.28). The value of the percentage error is calculated as: $5 \times 10^{-4} \times 33.36\% = 1.67 \times 10^{-7}$ deg/h$^{1/2}$. Thus the angle random walk coefficient for CIMU Z-axis gyro is estimated as: $Q = 5 \times 10^{-4} \pm 1.67 \times 10^{-7}$ deg/h$^{1/2}$. 

![Figure 3.15 Roll rate gyro Allan variance result with slope of -1](image)
Figure 3.16 Roll rate gyro Allan variance result with slope of -1/2

c) Drift rate ramp noise

In Figure (3.17), there is an example shown to obtain drift rate ramp noise coefficient from the Allan variance log-log plot. A straight line with slope of +1 is fitted to the long cluster time part of the plot and meets $T = 2^{1/2}$ hour line at a value of 1.5. The unit for velocity drift rate ramp is $\text{m/s/h}^2$. Inspection of the curve shows that the estimation percentage error in this region can reach 33.36% according to Equation (3.28). The value of the percentage error is equal to $1.5 \times 33.36\% = 0.5 \text{ m/s/h}^2$. Thus the drift rate ramp coefficient for CIMU Z-axis accelerometer is estimated as: $R = 1.5 \pm 0.5 \text{ m/s/h}^2$. 
Figures 3.15 - 3.17 shows the method of obtaining the various noise coefficients from the Allan variance plot. The predominant noise coefficients obtained for the three gyroscopes and three accelerometers are given in Table 3.2 and Table 3.3 respectively.

Table 3.3  Identified noise Coefficients for gyroscopes

<table>
<thead>
<tr>
<th>Gyroscopes</th>
<th>Quantization are second</th>
<th>Angle Random walk Deg/h (^{1/2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roll rate Gyro</td>
<td>5.4x10^{-4} ± 1.48x10^{-5}</td>
<td>5x10^{-4} ± 1.67x10^{-7}</td>
</tr>
<tr>
<td>Pitch rate Gyro</td>
<td>4.32x10^{-4} ± 1.16x10^{-5}</td>
<td>6x10^{-5} ± 2x10^{-3}</td>
</tr>
<tr>
<td>Yaw rate Gyro</td>
<td>3.6x10^{-5} ± 9.72x10^{-7}</td>
<td>2.7x10^{-5} ± 9x10^{-6}</td>
</tr>
</tbody>
</table>

Table 3.4  Identified noise coefficients for accelerometers

<table>
<thead>
<tr>
<th>Accelerometers</th>
<th>Quantization (m/h)</th>
<th>Velocity Random walk (m/s/√h)</th>
<th>Rate random walk (m/s/h/√h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>X Accelerometer</td>
<td>5.4x10^{-5} ± 1.458x10^{-6}</td>
<td>1.5x10^{-5} ± 5x10^{-6}</td>
<td>4.5 ± 1.5</td>
</tr>
<tr>
<td>Y Accelerometer</td>
<td>1.26x10^{-3} ± 3.4x10^{-5}</td>
<td>1.5x10^{-3} ± 5x10^{-4}</td>
<td>4 ± 1.33</td>
</tr>
<tr>
<td>Z Accelerometer</td>
<td>9x10^{-5} ± 2.43x10^{-6}</td>
<td>1.5x10^{-3} ± 5x10^{-4}</td>
<td>1.5 ± 0.5</td>
</tr>
</tbody>
</table>
3.12 WAVELET DECOMPOSITION FOR DE-NOISING INERTIAL SENSOR DATA

It is a well-known fact in inertial navigation that all gyro and accelerometer technologies suffer from relatively high frequency noise. The separation of the high and low frequency inertial sensor noise components can be done by de-noising the inertial measurements using Wavelet decomposition (Abdel-Hamid et al 2004, El-Diasty et al 2007).

3.12.1 Wavelets and Wavelet Transform (WT)

Wavelets, as a mathematical tool, are based on analyzing a signal through signal windowing but with variable window sizes. This gives an advantage to wavelet that it is capable of performing local analyses, i.e. analyzing a localized portion of a large signal. It is possible since wavelets allow the use of narrow windows (short time intervals) if high frequency information is needed and wide windows (long-time intervals) if low frequency information is required.

3.12.2 Wavelet Multiple-Level Of Decomposition (Wavelet Multi-Resolution Analysis)

In the implementation of the Discrete Wavelet Transform (DWT), the wavelet coefficients of a signal are computed by passing such a signal through two complementary half-band filters: a Low-Pass (LP) filter and a High-Pass (HP) filter. Therefore, the input signal will be decomposed into two parts. The first part will be the output of the HP filter (i.e. the details) while the second part will be the output of the LP filter (i.e. the approximation) as shown in Figure 3.18.
Based on the Nyquist theorem, if a signal has a sampling frequency of \( f_s \), the highest frequency component that the signal would represent is \( f_s/2 \). By applying the DWT to decompose a signal and recalling that the LP and HP filters (shown in the filter bank of Figure 3.18) have half-band characteristics, then the cutoff frequency of the LP filter is exactly at one half of the maximum frequency appearing at the signal. Hence, if the DWT is applied on an inertial data of sampling frequency \( f_s \), the approximation part will include those inertial signal components that have frequencies of less than \( f_s/4 \) while the details part will include the components of frequencies between \( f_s/4 \) and \( f_s/2 \).

In order to obtain finer resolution frequency components of a specific signal, the signal is broken down into many lower-resolution components by repeating the DWT decomposition procedure with successive decompositions of the obtained approximation parts. This procedure is called either wavelet multi-resolution analysis or wavelet multiple Level of Decomposition (LOD) or wavelet decomposition tree (Figure 3.19).

However, this capability of representing a signal at several levels of resolution constitutes one of the major powerful facilities of wavelets over other signal processing techniques. Using wavelet multi-resolution analysis, the signal can be represented by a finite sum of components at different

---

**Figure 3.18 Signal Decomposition by the Discrete Wavelet Transform (DWT)**

![Signal Decomposition Diagram](image)
resolutions, and hence, each component can be processed adaptively depending on the application at hand.

![Wavelet Decomposition Tree](image)

**Figure 3.19** Wavelet multi-resolution analysis considering three levels of decomposition (Wavelet Decomposition Tree)

Practically, an appropriate Level of Decomposition (LOD) is chosen based on the nature of the signal or on some specific criterion (like mean, covariance, probability density function etc.). Since noise is assumed to be zero mean, the wavelet decomposition level at which the mean becomes non-zero should be used for decomposition, since further decomposition would mean that actual trends in the data are being interpreted as noise. The Daubechies wavelet family provides the greatest degree of flexibility for parametric modifications involved in signal analysis.

### 3.12.3 Results and Discussion

The results of the simulation implemented using MATLAB for the modeled MEMS gyroscopes and accelerometers output with and without wavelet decomposition are shown in Figures 3.20 - 3.25. The Daubechies wavelet decomposition with the expected maximum wavelet LOD of three has been applied for all the three MEMS gyroscopes and accelerometer
simulated data. The improvement in the reduction of noise which is also quantified in terms of Signal-to-noise ratio (SNR) is given in Table 3.4

Figure 3.20 Wavelet denoising of Roll rate gyro raw output

Figure 3.21 Wavelet denoising of Pitch rate gyro raw output
Figure 3.22 Wavelet denoising of Yaw rate gyro raw output

Figure 3.23 Wavelet denoising of X Accelerometer raw output
Figures 3.20-3.25 clearly indicates that by using Wavelet decomposition the high frequency components of noise are eliminated while
the appropriate data of the inertial sensor contained in the low frequency component is retained.

Table 3.5 SNR values before and after Wavelet denoising

<table>
<thead>
<tr>
<th>Data points</th>
<th>SNR in dBs Before denoising</th>
<th>SNR in dBs After denoising</th>
<th>Improvement in SNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>4.2</td>
<td>17.726</td>
<td>13.526</td>
</tr>
<tr>
<td>2000</td>
<td>4.25</td>
<td>17.567</td>
<td>13.317</td>
</tr>
<tr>
<td>3000</td>
<td>4.4164</td>
<td>18.34</td>
<td>13.9236</td>
</tr>
<tr>
<td>4000</td>
<td>5.5495</td>
<td>19.105</td>
<td>13.5555</td>
</tr>
<tr>
<td>5000</td>
<td>4.09</td>
<td>18.818</td>
<td>14.728</td>
</tr>
</tbody>
</table>

The effectiveness of wavelet decomposition technique is realized by comparing it with a conventional low-pass filter and the results are shown in Figure 3.26 and the noise after the wavelet denoising and low pass filter are shown in Figure 3.27

Figure 3.26 Wavelet denoising and low-pass filter output of pitch rate MEMS gyroscope raw output.
From the Figures 3.20-3.25, it is observed that using wavelet decomposition, the high frequency noise present in the inertial sensors are removed and the low frequency signal which contains original signal is obtained. The errors obtained using the original raw MEMS Gyroscopes and Accelerometers data (before de-noising) as well as the wavelet de-noised data were computed. The improvement in the reduction of noise which is also quantified in terms of Signal-to-noise ratio (SNR) is given in Table 3.4 and the effectiveness of wavelet decomposition technique compared to a conventional low-pass filter is shown in Figure 3.26 and Figure 3.27. It is evident that significant reduction in the measurement noise was achieved, thus reducing the measurement uncertainty. The de-noising procedure was specifically beneficial in improving the SNR of pitch rate MEMS gyroscope measurement. Wavelet-based denoising methods have the advantage over low-pass filtering in that relevant detail information is retained, while small details due to noise are discarded. MEMS sensor output denoised using wavelet decomposition improves the SNR which in turn increase the accuracy of the navigation.
3.13 SIMULATION OF STRAPOWN INERTIAL NAVIGATION SYSTEM (SDINS) BASED ON QUATERNION APPROACH

A strap-down inertial navigation system uses orthogonal accelerometers and gyro triads rigidly fixed to the axes of the moving vehicle. The angular motion of the system is continuously measured using the rate sensors. The gyros measure the vehicle body rates and the accelerometers measure specific force in vehicle body frame. The navigation computer transforms the measured acceleration analytically or mathematically to the desired navigation frame. Then INS mechanization (Itzhack and Bar-Itzhack 1977, George Siouris 1993, Greenspan 1995, Minoru 1986) is used to find position, velocity and attitude.

3.13.1 Reference Frames and Transformations

a) The inertial frame (i-frame) has its origin at the centre of the Earth and axes which are non-rotating with respect to the fixed stars with its z-axis parallel to the spin axis of the Earth, x-axis pointing towards the mean vernal equinox, and y-axis completing a right handed orthogonal frame.

b) The Earth frame (e-frame) has its origin at the centre of mass of the Earth and axes which are fixed with respect to the Earth. Its x-axis points towards the mean meridian of Greenwich, z-axis is parallel to the mean spin axis of the Earth, and y-axis completes a right-handed orthogonal frame as shown in Figure 3.28 (A).

c) The navigation frame (n-frame) is a local geodetic frame which has its origin coinciding with that of the sensor frame, and axes with x-axis pointing towards geodetic north, z-axis orthogonal to the reference ellipsoid pointing down, and y-axis completing a right-handed orthogonal frame, i.e. the north-east-down (NED) system as shown in Figure 3.28 (A).
Figure 3.28  (A) Relations between ECEF-frame (e), local geodetic-frame (t) and Inertial-frame (i), (B) Body frame

d) The body frame (b-frame) is an orthogonal axis set which is aligned with the roll, pitch and heading axes of a vehicle, i.e. forward-transversal-down as shown in Figure 3.28(B).

3.14 STRAPDOWN INS COMPUTATION

Figure 3.29 Strapdown INS architecture

Figure 3.29 shows the Strapdown INS (SDINS) architecture. It consists of IMU, Attitude computation and Navigation computation (Sonmez
IMU comprises of three gyroscopes and three accelerometers which measure angular rates and accelerations along three orthogonal axes respectively. Attitude computation calculates Euler angle and Navigation computation calculates NED velocities and Latitude, Longitude and Altitude positions.

### 3.15 INERTIAL NAVIGATION EQUATIONS

The position in the n-frame is expressed by curvilinear coordinates are given in Equation (3.29).

\[ r_n = (\varphi \quad \lambda \quad h)^T \] (3.29)

and the velocities in the n–frame are defined by Equation (3.30)

\[
V_n = 
\begin{pmatrix}
V_N \\
V_E \\
V_D
\end{pmatrix} =
\begin{pmatrix}
M + h & 0 & 0 \\
0 & (N + h)\cos \varphi & 0 \\
0 & 0 & -1
\end{pmatrix}
\] (3.30)

### 3.16 SDINS MECHANIZATION USING QUATERNION

Usually quaternion implementation is preferred (Minoru 1986, Mohamed et al 2005, Itzhack and Bar-Itzhack 1997, Kim and Golnaraghi 2004) in updating the attitude as the linearity of the quaternion differential equations, the lack of trigonometric functions, and the small number of parameters allows efficient implementation. The quaternion attitude representation is a four-parameter representation based on the idea that a transformation from one coordinate frame to another may be effected by a single rotation about a vector \( \mu \).
A quaternion is a four–element vector as given in Equation (3.31)

\[
q = \begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{pmatrix} = \begin{pmatrix}
\frac{\mu_x \sin \left( \frac{\mu}{2} \right)}{\mu} \\
\frac{\mu_y \sin \left( \frac{\mu}{2} \right)}{\mu} \\
\frac{\mu_z \sin \left( \frac{\mu}{2} \right)}{\mu} \\
\cos \left( \frac{\mu}{2} \right)
\end{pmatrix}
\]  

(3.31)

where \( \mu_x, \mu_y, \mu_z \) are components of the rotation angle vector \( \mu \) and

\[
\mu = (\mu_x^2 + \mu_y^2 + \mu_z^2)^{1/2}
\]

The quaternions should satisfy the normality condition as given in Equation (3.32)

\[
q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1
\]  

(3.32)

The differential equations for the quaternion parameters is given by Equation (3.33).

\[
q = \frac{1}{2} \begin{pmatrix}
0 & \omega_z & -\omega_y & \omega_x \\
-\omega_z & 0 & \omega_x & \omega_y \\
\omega_y & -\omega_x & 0 & \omega_z \\
-\omega_x & -\omega_y & -\omega_z & 0
\end{pmatrix} \dot{q}
\]  

(3.33)

where \( \omega = (\omega_x, \omega_y, \omega_z) \) is the angular velocity of the body rotation. The transformations between the quaternion and the DCM are \( C^n_b \) accomplished by Equation (3.34)

\[
C^n_b = \begin{pmatrix}
(q_i^2 - q_5^2 - q_5^2 + q_i^2) & 2(q_iq_2 - q_iq_4) & 2(q_iq_3 - q_iq_4) \\
2(q_iq_2 - q_iq_4) & (q_i^2 - q_i^2 - q_3^2 + q_4^2) & 2(q_2q_3 - q_2q_4) \\
2(q_iq_3 - q_iq_4) & 2(q_2q_3 - q_2q_4) & (q_i^2 - q_i^2 - q_2^2 + q_3^2)
\end{pmatrix}
\]  

(3.34)
3.16.1 Attitude Integration

The attitude dynamics are defined by Equation (3.35 to 3.37)

When $\Omega$ represents the skew symmetric matrix form of the vector $\omega$ and $\Omega_{\text{en}}^b$ is the outputs of the strapdown gyros, then $\Omega_{\text{en}}^b$ is obtained by the Equation (3.35)

$$\Omega_{\text{en}}^b = \Omega_{\text{ib}}^b - C_n^b \left[ \Omega_{\text{ic}}^n + \Omega_{\text{en}}^n \right]$$  \hspace{1cm} (3.35)

where $\Omega_{\text{ic}}^n = \begin{bmatrix} \Omega \cos \phi & 0 & -\Omega \sin \phi \end{bmatrix}^T$  \hspace{1cm} (3.36)

and $\Omega_{\text{en}}^n = \begin{bmatrix} V_E \in (R, h) + h \in (R, h) + h \in (R, h) \frac{V_N}{(R, h) + h} \frac{V_E \tan \phi}{(R, h) + h} \end{bmatrix}^T$  \hspace{1cm} (3.37)

The body angular increment with respect to the navigation frame are obtained by Equation (3.38)

$$\Delta \theta_{\text{ib}}^b = \left( \Delta \theta_x, \Delta \theta_y, \Delta \theta_z \right)^T \Delta \theta_{\text{ib}}^b - C_n^b \left( \omega_{\text{ic}}^n + \omega_{\text{en}}^n \right) \Delta t$$  \hspace{1cm} (3.38)

and the magnitude of the angular increment is calculated by Equation (3.39)

$$\Delta \theta = \sqrt{\Delta \theta_x^2 + \Delta \theta_y^2 + \Delta \theta_z^2}$$  \hspace{1cm} (3.39)

The angular increments obtained in Equations (3.38) and (3.39) are used to update the quaternion

$$q_{k+1} = q_k + 0.5 \begin{bmatrix} c & s \Delta \theta_x & -s \Delta \theta_y & s \Delta \theta_z \\ -s \Delta \theta_z & c & s \Delta \theta_x & s \Delta \theta_y \\ s \Delta \theta_y & -s \Delta \theta_x & c & s \Delta \theta_z \\ -s \Delta \theta_x & s \Delta \theta_y & -s \Delta \theta_z & c \end{bmatrix} q_k$$  \hspace{1cm} (3.40)
where
\[ s = \frac{2}{\Delta \theta} \sin \frac{\Delta \theta}{2} \left( 1 - \frac{\Delta \theta^2}{24} + \frac{\Delta \theta^4}{1920} + \ldots \right) \]
\[ c = 2 \left( \cos \frac{\Delta \theta}{2} - 1 \right) = \frac{\Delta \theta^2}{4} + \frac{\Delta \theta^4}{192} + \ldots \]

### 3.16.2 Velocity and Position Integration

The body frame velocity increment due to the specific force is transformed to the navigation frame is given in Equation (3.41):

\[ \dot{V}_n = C_n^b \dot{f}^b - (2 \omega_{ie}^n + \omega_{en}^n) \times V_n + \gamma^b \Delta t \]  
(3.41)

where, \( \dot{f}^b \) is the output of the accelerometers and represents the specific forces in body axes; \( \omega_{ie}^n \) and \( \omega_{en}^n \) are the turn rates of the Earth frame (e-frame) with respect to the inertial frame (i-frame) and the n-frame to the e-frame in the n-frame respectively; \( \gamma \) is the local gravity.

\[ \gamma^n = (0 \ \Theta \ \gamma)^T \]  
(3.42)

where \( \gamma^n \) is the normal gravity at the geodetic latitude \( \varphi \) and ellipsoidal height \( h \) as given in Equation (3.43).

\[ \gamma = a_1(1 + a_2 \sin^2 \varphi + a_3 \sin^4 \varphi) + (1 + a_2 \sin^2 \varphi)h + a_6 h^2 \]  
(3.43)

Using WGS84 Gravity Model:

\[ a_1 = 9.7803267715 \quad a_4 = -0.0000030876910891 \]
\[ a_2 = 0.0052790414 \quad a_5 = 0.0000000043977311 \]
\[ a_3 = 0.0000232718 \quad a_6 = 0.000000000007211 \]
The velocity integration can be performed as given in Equation (3.44)

\[ V_{k+1}^n = V_k^n + \Delta V_{k+1}^n \]  

(3.44)

and the positions are integrated using the second order Runge-Kutta method:

\[ r_{k+1}^n = r_k^n + 0.5 \, r^0 \, (V_k^n + \Delta V_{k+1}^n) \Delta t \]  

(3.45)

where M and N are the radii of curvature in the meridian and prime vertical, respectively. Figure 3.30 summarizes the overall navigation frame SDINS mechanization described in this chapter:

**Figure 3.30** Strapdown inertial navigation systems (SDINS) mechanization

### 3.16.3 Results and Discussions

Using the quaternion algorithm discussed in this chapter, the SDINS simulation for attitude, velocity and position are carried out for various trajectories and also the simulated data is compared with the real time data for validation in Figure 3.31-3.33, for the trajectory shown in Figure 2.2 which is used in the previous chapter to validate the sensor data generation
algorithm Figures 3.31-3.33 show the comparison of simulated and real attitude, velocity and position.

Figure 3.31 Comparison of simulated and original attitude angles

Figure 3.32 Comparison of simulated and original NED velocities
It is found that from the Figures 3.31-3.33, the simulated results almost follow the real data and the quantitative analysis of the position error between the simulated and real data is given in Table 3.6.

**Table 3.6 Quantitative analysis of position error between the simulated and real time data**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MSE</th>
<th>RMSE</th>
<th>STD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Latitude(deg)</td>
<td>2.8*10^-5</td>
<td>0.0053</td>
<td>0.0057</td>
</tr>
<tr>
<td>Longitude(deg)</td>
<td>2.2*10^-5</td>
<td>0.0046</td>
<td>0.013</td>
</tr>
<tr>
<td>Altitude(m)</td>
<td>5.3*10^-5</td>
<td>0.0072</td>
<td>0.0087</td>
</tr>
</tbody>
</table>

Hence from the Table 3.6, it is observed that the position accuracy is better. Figure 3.34 – 3.37 shows the position, attitude and velocity estimation for various trajectories with the MEMS inertial sensor data generated using the data generation algorithm and added with the modeled noises.

**Figure 3.33 Comparison of simulated and original position**

It is found that from the Figures 3.31-3.33, the simulated results almost follow the real data and the quantitative analysis of the position error between the simulated and real data is given in Table 3.6.

**Table 3.6 Quantitative analysis of position error between the simulated and real time data**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MSE</th>
<th>RMSE</th>
<th>STD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Latitude(deg)</td>
<td>2.8*10^-5</td>
<td>0.0053</td>
<td>0.0057</td>
</tr>
<tr>
<td>Longitude(deg)</td>
<td>2.2*10^-5</td>
<td>0.0046</td>
<td>0.013</td>
</tr>
<tr>
<td>Altitude(m)</td>
<td>5.3*10^-5</td>
<td>0.0072</td>
<td>0.0087</td>
</tr>
</tbody>
</table>

Hence from the Table 3.6, it is observed that the position accuracy is better. Figure 3.34 – 3.37 shows the position, attitude and velocity estimation for various trajectories with the MEMS inertial sensor data generated using the data generation algorithm and added with the modeled noises.
Figure 3.34 Navigation parameters of Straight flight Trajectory

Figure 3.35 Navigation parameters of Change in Heading and Altitude Trajectory
Figure 3.36 Navigation parameters of 180 degree turn Trajectory

Figure 3.37 Navigation parameters of 180 degree turn with Pitch Up Trajectory
In order to emphasize the need for the calibration of the inertial navigation, a typical plot showing the deviation of the position estimation of SDINS algorithm for the original trajectory is shown in Figure 3.38. The simulated trajectory shown in Figure 3.38 is actually obtained from the original trajectory shown in the same figure, using the sensor data generation algorithm discussed in the previous chapter and then modeled with the sensor errors discussed in this chapter and finally the position estimation is carried out using the quaternion based navigation algorithm.

![6 DOF Trajectory](image)

**Figure 3.38  Comparison of trajectory generated using simulated sensor data and original trajectory**

From Figure 3.38, it is clear that when the navigation algorithm is carried out with the simulated data (which actually mimics the real sensor data), the trajectory generated by the simulated data deviates from the original trajectory as time prolongs is shown which is already mentioned as the main disadvantages of the inertial navigation.
3.17 CONCLUSION

In this chapter, the need for the going on to the MEMS inertial sensors is discussed followed by the types and errors of the MEMS inertial sensors. The deterministic and random errors of the inertial sensors are mathematical modeling and added to the raw sensor data generated for a given trajectory using the data generation algorithm discussed in the previous chapter. Results show that the simulated sensor data resembles the real sensor data. Then the data is analyzed using Allan variance technique to identify the various random error components present in the inertial sensors and listed in Tables 3.2 and 3.3. Wavelet denoising technique is applied to de-noise the sensor data and the performance of it is highlighted by comparing it with a conventional low pass filter. Finally, the simulation of strapdown inertial navigation system based on quaternion approach is carried out. The improvement in the position accuracy with and without the denoising is quantitatively analyzed and shown in terms of MSE, RMSE and STD in Table 3.6. Finally, the deviation of the trajectory generated using simulated sensor data from the original trajectory is shown in Figure 3.38 to emphasize the need for the calibration. Hence, vision navigation data is used to calibrate the SDINS sensor errors which is dealt in detail in the subsequent chapter.