Chapter-IV

Patterned Constructions of Partially Balanced Incomplete Block Designs by Juxtaposing Association Matrices

4.1 Introduction

The main problem in design of experiment is to test the performance of treatments by developing efficient designs having minimum error under the given constraints. In such situation, a large number of methods of construction of partially balanced incomplete block designs are available in literature. One of the methods of constructing partially balanced incomplete block designs is the patterned construction. Construction of partially balanced incomplete block designs with two, three & four associate classes by the method of new juxtaposition patterns have been discussed in literature by various authors such as Clatworthy (1973), John and Turner (1977), Bhagwandas & Parihar (1980, 1982), Bhagwandas et al (1985), Kageyama and Mohan (1985), Puri et al (1987), Kageyama and Bhagwandas (1987), Banerjee and Kageyama (1989) etc. Recently, Kageyama and Sinha (2003) and Garg (2005) have constructed partially balanced incomplete block designs with the use of juxtaposition patterns of the incidence matrices.

First of all, we define some terms which have been generally used in the construction of designs as:
**Incidence Matrix:** Associated with any design $D$, the matrix $N = (n_{ij})$, $(i = 1, 2, \ldots, v; j = 1, 2, \ldots, b)$ is known as incidence matrix of design $D$; where $n_{ij}$ denotes the number of times the $i^{th}$ treatment occurs in the $j^{th}$ block.

i.e.

$$N = \begin{bmatrix}
n_{11} & n_{12} & \cdots & n_{1b} \\
n_{21} & n_{22} & \cdots & n_{2b} \\
\vdots & \vdots & \ddots & \vdots \\
n_{v1} & n_{v2} & \cdots & n_{vb}
\end{bmatrix}$$

For a binary design

$$n_{ij} = \begin{cases} 
1 & \text{if } i^{th} \text{ treatment occurs in } j^{th} \text{ block} \\
0 & \text{otherwise}
\end{cases}$$

**Association Matrix:** The $i^{th}$ association matrix $B_i$; $i = 0, 1, \ldots, m$ of an association scheme is a symmetric matrix of order $v$ with elements $b'^{i}_{\alpha\beta}$. Thus by the definition, we have

$$B_i = \begin{bmatrix}
b_{11}^i & b_{12}^i & \cdots & b_{1v}^i \\
b_{21}^i & b_{22}^i & \cdots & b_{2v}^i \\
\vdots & \vdots & \ddots & \vdots \\
b_{vi}^i & b_{v2}^i & \cdots & b_{vv}^i
\end{bmatrix}$$
Where

\[ b_{\alpha \beta}^{i} = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are mutually } i^{th} \text{ associates,} \\ 0 & \text{otherwise; } i = 1, 2, \ldots, m; \quad \alpha, \beta = 1, 2, \ldots, v \end{cases} \]

In the present chapter, starting from the association matrices of a group divisible partially balanced incomplete block design with two associate classes and following certain juxtaposition pattern involving a type of matrix in which all elements are unity and complement of the association matrices, we have obtained some new partially balanced incomplete block designs with two, three and four associate classes.

In section 4.2, association schemes of two associate class group divisible and some new three associate and four associate classes partially balanced incomplete block designs along with their parameters are given and section 4.3 includes general analysis of partially balanced incomplete block designs with m-associate classes and its complete analysis with two associate classes. Methods of construction of regular group divisible, singular group divisible, three associate and four associate class partially balanced incomplete block designs along with respective numerical illustrations have been discussed in section 4.4.

In section 4.5, the designs corresponding to different values of parameters developed in theorems 4.4.1 to 4.4.6 have been listed in the tables 4.5.1 to 4.5.4. The symbol ‘E’ denotes the average efficiency factor of the partially balanced incomplete block designs with respect to randomized block design (RBD), ‘E_i’ denotes the efficiency factor of the treatment contrasts which are i^{th} (i = 1, 2, \ldots, m) associates with respect to randomized block designs.
Some of the regular group divisible designs, singular group divisible designs, three associate and four associate class partially balanced incomplete block designs constructed in this chapter are new in the sense that these designs are not listed in Clatworthy (1973), Freeman (1976), John and Turner (1977), Parihar (1980, 1982), Bhagwandas et al (1985), Banerjee and Kageyama (1988), Kageyama et al (1989) and Kageyama & Mohan (1985). For the complete description of PBIB designs along with their association schemes, we refer to Raghavarao (1971) and Dey (2010). The new PBIB designs constructed here are desirable ones and of much statistical interest because of their good efficiencies.

As for as non-existence of the newly constructed designs is concerned, it has been numerically verified (with the help of blue-bit, online computer software) that all the characteristic roots of the designs listed in tables 4.5.1, 4.5.2, 4.5.3 & 4.5.4 are non-negative. Hence, the necessary condition for the existence of partially balanced incomplete block designs with m-associate classes is satisfied. The well known result [see Dey (2010)] is that the characteristics roots $\theta_i \ (i = 0, 1, ..., m)$ of $NN'$ matrix of any partially balanced incomplete block design should be non-negative.
4.2 Association Schemes of Group Divisible and New Three & Four Associate Class Partially Balanced Incomplete Block Designs

In this section, association schemes of group divisible and new three associate and four associate classes partially balanced incomplete block designs are defined as follows:

4.2.1 Association scheme of group divisible designs: Let there are \( v = mn \) treatments (\( m, n \) integers; \( m \geq 2, n \geq 2 \)) arranged in rectangular array with \( m \) rows and \( n \) columns. On these \( v \) treatments, we define a group divisible association scheme as: two treatments are first associates if they belong to the same row of the array and are second associates otherwise.

The parameters of the group divisible association scheme are as under:

\[
\begin{align*}
\nu &= mn, \quad n_1 = n - 1, \quad n_2 = n(m - 1); \\
P_1 &= \left[ p^{1}_{ij} \right] = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix} \\
P_2 &= \left[ p^{2}_{ij} \right] = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}
\end{align*}
\]

It can be easily verified that for group divisible designs the characteristics roots of \( NN^t \) matrix other than \( r_k \) are \( r - \lambda_1 \) and \( r - n\lambda_2 \) with respective multiplicities \( m(n - 1) \) and \( m - 1 \). Making use of the values of the characteristic roots, group divisible designs are further subdivided into three classes by Bose and Connor (1952).
1. Singular (S) if \( r - \lambda_1 = 0 \)

2. Semi-regular (SR) if \( r - \lambda_1 > 0 \) and \( rk - v\lambda_2 = 0 \)

3. Regular (R) if \( rk - v\lambda_2 > 0 \)

4.2.2 Association scheme of new three associate classes partially balanced incomplete block designs: This association scheme is based on two sets each having \( mn \) number of treatments. Any two treatments in same row in a set are the first associates, whereas any two treatments in different rows in the same set are the second associates. Any two treatments occurring in different sets are the third associates.

The parameters of the association scheme are as under:

\[
v = 2mn, \ n_1 = n - 1, \ n_2 = n(m - 1), \ n_3 = mn
\]

\[
P_1 = [p_{ij}^1] = \begin{bmatrix} n-2 & 0 & 0 \\ 0 & n(m-1) & 0 \\ 0 & 0 & mn \end{bmatrix}
\]

\[
P_2 = [p_{ij}^2] = \begin{bmatrix} 0 & n-1 & 0 \\ n-1 & n(m-2) & 0 \\ 0 & 0 & mn \end{bmatrix}
\]

\[
P_3 = [p_{ij}^3] = \begin{bmatrix} 0 & 0 & n-1 \\ 0 & 0 & n(m-1) \\ n-1 & n(m-1) & 0 \end{bmatrix}
\]
4.2.3 Association scheme of new three associate classes partially balanced incomplete block designs: This association scheme is based on three sets each having \(mn\) number of treatments. Any two treatments in same row in a set are the first associates, whereas any two treatments in different rows in the same set are the second associates. Any two treatments occurring in different sets are the third associates.

The parameters of the association scheme are as under:

\[ v = 3mn, \quad n_1 = n - 1, \quad n_2 = n(m - 1), \quad n_3 = 2mn \]

\[
P_1 = \begin{bmatrix} p_{1,1}^{ij} \\ \end{bmatrix} = \begin{bmatrix} n - 2 & 0 & 0 \\ 0 & n(m - 1) & 0 \\ 0 & 0 & 2mn \end{bmatrix}
\]

\[
P_2 = \begin{bmatrix} p_{2,1}^{ij} \\ \end{bmatrix} = \begin{bmatrix} 0 & n - 1 & 0 \\ n - 1 & n(m - 2) & 0 \\ 0 & 0 & 2mn \end{bmatrix}
\]

\[
P_3 = \begin{bmatrix} p_{3,1}^{ij} \\ \end{bmatrix} = \begin{bmatrix} 0 & 0 & n - 1 \\ 0 & 0 & n(m - 1) \\ n - 1 & n(m - 1) & mn \end{bmatrix}
\]

4.2.4 Association scheme of new four associate classes partially balanced incomplete block designs: This association scheme is also based on two sets each having \(mn\) number of treatments. Any two treatments in same row in a set are the first associates, whereas any two the treatments in different rows of the same set are second associates. Further, any
two treatments occurring in the different sets in the corresponding row of first associates
are third associates and any two treatments occurring in different rows of the different
sets are the fourth associates.

The parameters of the association scheme are as under:

\[ v = 2mn, \ n_1 = n - 1, \ n_2 = n(m - 1), \ n_3 = mn \]

\[
P_1 = \left[ p_{ij}^1 \right] = \begin{bmatrix}
    n - 2 & 0 & 0 & 0 \\
    0 & n(m - 1) & 0 & 0 \\
    0 & 0 & n & 0 \\
    0 & 0 & 0 & n(m - 1)
\end{bmatrix}
\]

\[
P_2 = \left[ p_{ij}^2 \right] = \begin{bmatrix}
    0 & n - 1 & 0 & 0 \\
    n - 1 & n(m - 2) & 0 & 0 \\
    0 & 0 & 0 & n \\
    0 & 0 & n & n(m - 2)
\end{bmatrix}
\]

\[
P_3 = \left[ p_{ij}^3 \right] = \begin{bmatrix}
    0 & 0 & n - 1 & 0 \\
    0 & 0 & 0 & n(m - 1) \\
    n - 1 & 0 & 0 & 0 \\
    0 & n(m - 1) & 0 & 0
\end{bmatrix}
\]

\[
P_4 = \left[ p_{ij}^4 \right] = \begin{bmatrix}
    0 & 0 & 0 & n - 1 \\
    0 & 0 & n & n(m - 2) \\
    0 & n & 0 & 0 \\
    n - 1 & n(m - 2) & 0 & 0
\end{bmatrix}
\]
4.3 Analysis of Partially Balanced Incomplete Block Designs

In this section, first we will discuss the general analysis with m-associate classes and then find out the solution of normal equations from the analysis of partially balanced incomplete block designs with two associate classes. We have obtained two types of efficiencies and an average efficiency factor of designs with two associate classes. These efficiencies are important from the point of view of comparisons of designs.

4.3.1 Analysis of Partially Balanced Incomplete Block Designs with m-Associate Classes:

Let D be a PBIB design with parameters v, b, r, k, $\lambda_i$, $n_i$, $p_{jk}^i$; i, j, k = 1, 2, ..., m. if $\tau_i$ is the effect of the ith treatment (i = 1, 2, ..., v), the reduced normal equations, using intra-block information, for treatment effects are given by

$$r(k-1)\tau_i - \lambda_1 S_1(\tau_i) - \lambda_2 S_2(\tau_i) - ... - \lambda_m S_m(\tau_i) = kQ_i$$  \hspace{1cm} (4.3.1)

Where i = 1, 2, ..., v and $S_j(\tau_i)$ denotes the sum of those $n_j$ treatments effects which are the $j^{th}$ associates of i, for j = 1, 2, ..., m.

Let the $n_j$, $j^{th}$ associates of i have treatment effects $\alpha_{1}^i, \alpha_{2}^i, ..., \alpha_{n_j}^i$ . Rewriting the normal equations in respect of $\alpha_{1}^i, \alpha_{2}^i, ..., \alpha_{n_j}^i$, we get

$$r(k-1)\alpha_{1}^i - \lambda_1 S_1(\alpha_{1}^i) - \lambda_2 S_2(\alpha_{1}^i) - ... - \lambda_m S_m(\alpha_{1}^i) = kQ_i(\alpha_{1}^i)$$

$$r(k-1)\alpha_{2}^i - \lambda_1 S_1(\alpha_{2}^i) - \lambda_2 S_2(\alpha_{2}^i) - ... - \lambda_m S_m(\alpha_{2}^i) = kQ_i(\alpha_{2}^i)$$

$$...$$

$$r(k-1)\alpha_{n_j}^i - \lambda_1 S_1(\alpha_{n_j}^i) - \lambda_2 S_2(\alpha_{n_j}^i) - ... - \lambda_m S_m(\alpha_{n_j}^i) = kQ_i(\alpha_{n_j}^i)$$  \hspace{1cm} (4.3.2)
Summing the \( n_j \) equations in (4.3.2), we get an equation of the following form:

\[
(r - 1)S_j(\tau_j) - \lambda_1 P_1 - \lambda_2 P_2 - \ldots - \lambda_m P_m = kS_j(Q_j)
\]  
(4.3.3)

Where \( P_1, P_2, \ldots, P_m \) are linear functions of treatments effects and \( S_j(Q_j) \) is defined analogously as \( S_j(\tau_j) \).

It is easy to verify, from the definition of PBIB design, that

\[
P_u = \sum_{i=1}^{m} p_{ju}' S_i(\tau_i)
\]
for \( u \neq j, \ u = 1, 2, \ldots, m \)

\[
P_j = n_j \tau_i + \sum_{i=1}^{m} p_{ji}' S_i(\tau_i)
\]
(4.3.4)

Substituting for \( P_u \)'s in (4.3.3) and rearranging, we get the following equation:

\[
r(k - 1)S_j(\tau_i) - S_j(\tau_i) \sum_{i=1}^{m} \lambda_i p_{ji}' - S_2(\tau_i) \sum_{i=1}^{m} \lambda_i p_{ji}^2 - \ldots - S_m(\tau_i) \sum_{i=1}^{m} \lambda_i p_{ji}^m
\]

\[-n_j \lambda_j \tau_i = kS_j(Q_j)
\]
for \( j = 1, 2, \ldots, m \).  
(4.3.5)

These equations can be solved conveniently for \( \tau_i \) by taking the side restriction:

\[
\sum_{i=1}^{m} \tau_i = \tau_i + S_1(\tau_i) + S_2(\tau_i) + \ldots + S_m(\tau_i) = 0
\]  
(4.3.6)

\[\textbf{4.3.2 Solution of Normal Equations from the General Analysis of the Partially Balanced Incomplete Block Designs for Two Associate Class (m=2)}\]

Put \( m = 2 \) in the equation (4.3.1), we get

\[
r(k - 1)\tau_i - \lambda_1 S_1(\tau_i) - \lambda_2 S_2(\tau_i) = kQ_i
\]  
(4.3.7)
Similarly, (4.3.5) reduces to following equations:

\[
r(k-1)S_1(\tau_i) - S_1(\tau_i)(\lambda_1 p_{11} + \lambda_2 p_{12}) - n_1 \lambda_1 \tau_i
- S_2(\tau_i)\left(\lambda_4 p_{11}^2 + \lambda_2 p_{12}^2\right) = kS_1(Q_i) 
\]  \hspace{1cm} (4.3.8)

\[
r(k-1)S_2(\tau_i) - S_1(\tau_i)(\lambda_1 p_{12} + \lambda_2 p_{22}) - n_2 \lambda_2 \tau_i
- S_2(\tau_i)(\lambda_1 p_{12} + \lambda_2 p_{22}) = kS_2(Q_i) 
\]  \hspace{1cm} (4.3.9)

Eliminating \( S_2(\tau_i) \) from (4.3.7) and (4.3.8), using the “restriction” (4.3.6), and simplifying, we get

\[
\{r(k - 1) + \lambda_2\} \tau_i - (\lambda_1 - \lambda_2)S_1(\tau_i) = kQ_i
(\lambda_2 - \lambda_1)p_{12}^2 \tau_i + S_1(\tau_i)(r(k-1) + \lambda_2 + (\lambda_2 - \lambda_1)(p_{11}^1 - p_{11}^2)) = kS_1(Q_i) 
\]

Let us define the following constants:

\[
A_1 = r(k-1) + \lambda_2 \\
A_2 = \lambda_2 - \lambda_1 \\
B_1 = (\lambda_2 - \lambda_1)p_{12}^2 \\
B_2 = r(k-1) + \lambda_2 + (\lambda_2 - \lambda_1)(p_{11}^1 - p_{11}^2) 
\]  \hspace{1cm} (4.3.10)

Using these constants, one gets a solution for \( \tau_i \) as:

\[
\hat{\tau}_i = \frac{k(B_2Q_i - A_2S_1(Q_i))}{(A_1B_2 - A_2B_1)}, \quad i = 1, 2, \ldots, v. 
\]  \hspace{1cm} (4.3.11)

The variance of an estimated elementary contrast among treatment effects is given by:
\[ \text{Var}(\hat{r}_i - \hat{r}_j) = \frac{2k(B_2 + A_2)\sigma^2}{A_1B_2 - A_2B_1} = v_1 \text{ (say)}, \] if i and j are first associates; \hspace{1cm} (4.3.12)

\[ = \frac{2kB_2\sigma^2}{A_1B_2 - A_2B_1} = v_2 \text{ (say)}, \] if i and j are second associates. \hspace{1cm} (4.3.13)

The average variance of all the treatment contrasts is given by

\[ AV_r = \frac{n_1v_1 + n_2v_2}{n_1 + n_2} \hspace{1cm} (4.3.14) \]

Hence, the efficiency factor of two kinds of comparison and the average efficiency factor are given by

\[ E_1 = \frac{A_1B_2 - B_2A_1}{kr(B_2 + A_2)}, \text{ efficiency factor of the first associates w.r.t. RBD;} \]

\[ E_2 = \frac{A_1B_2 - B_2A_1}{krB_2}, \text{ efficiency factor of the second associates w.r.t. RBD;} \]

\[ E = \frac{A_1B_2 - B_2A_1}{kr[B_2(v-1) + n_1A_2]}, \text{ average efficiency factor w.r.t. RBD} \]

4.4 Construction of Group Divisible, Three Associate & Four Associate Classes Partially Balanced Incomplete Block Designs

In this section, there are six theorems from which we obtain some regular group divisible, singular group divisible, three associate and four associate classes PBIB designs with their
respective association schemes defined in section 4.2. Theorem 4.4.1 yields regular group divisible designs, Theorem 4.4.2 yields singular group divisible designs; Theorem 4.4.3, Theorem 4.4.4 & Theorem 4.4.5 yields designs with three associate classes following new association scheme defined in section 4.2.2 & 4.2.3 and Theorem 4.4.6 yields designs with four associate class following new association scheme defined in section 4.2.4. All the theorems have also been illustrated numerically.

**Theorem 4.4.1:** Let $B_0$, $B_1$ and $B_2$ denote the association matrices of a partially balanced incomplete block design with two associate classes following group divisible association scheme with $m$ rows each of size $n$. Consider the following structure

$$
N^* = \begin{bmatrix}
\overline{B}_1 & J \\
J & \overline{B}_1
\end{bmatrix}
$$

Then $N^*$ becomes the incidence matrix of a symmetric regular group divisible type PBIB design with parameters $v^* = b^* = 2mn$, $r^* = k^* = n(2m - 1) + 1$, $\lambda_1 = r^* - 1$, $\lambda_2 = 2[n(m-1)+1]$, $n_1 = n - 1$, $n_2 = \lambda_1, m^* = 2m, n^* = n$.

Here $J$ is a $(mn) \times (mn)$ type matrix of ones and $\overline{B}_1$ is obtained from $B_1$ by interchanging zero’s and one’s with $m \geq 2, n \geq 2$.

**Proof:** Since $N^*$ denotes the incidence matrix of a symmetric regular group divisible design containing only zeros and ones $N^* N^{*\prime}$ can be written as:

$$
N^* N^{*\prime} = \begin{bmatrix}
\overline{B}_1 \overline{B}_1' + JJ' & \overline{B}_1 J' + J \overline{B}_1' \\
J \overline{B}_1' + \overline{B}_1 J' & JJ' + \overline{B}_1 \overline{B}_1'
\end{bmatrix}
$$
and it can be treated as the treatment structure matrix of a new design D in which rows correspond to treatments and columns correspond to blocks. Since the row sum and column sum of $N^*$ is $[n(2m - 1)+1]$, so each treatment in D is replicated $[n(2m - 1)+1]$ times and each block contains $[n(2m - 1)+1]$ treatments. Because $N^*N^*$ is the treatment structure matrix of D, its form shows that treatments can be partitioned into two groups. Taking this partition of treatments as groups of a GD designs, we found that

$$\lambda_1 = r^*-1, \lambda_2 = 2[n(m-1)+1], n_1 = n - 1, n_2 = \lambda_1, m^* = 2m, n^* = n,$$

which shows that $r^* > \lambda_1$ and $r^*k^*-v^*\lambda_2 > 0$, hence the design is a regular group divisible design.

**Illustration 4.4.1:** For $m = 2, n = 3$ in the design parameters defined in Theorem 4.4.1, the structure

$$N^* = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$
becomes the incidence matrix of a regular group divisible design with parameters $v^* = b^* = 12, r^* = k^* = 10, \lambda_1 = 9, \lambda_2 = 8, n_1 = 2, n_2 = 9, m^* = 4, n^* = 3$, the blocks of which are:

1. $\{1, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
2. $\{2, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
3. $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
4. $\{1, 2, 3, 4, 7, 8, 9, 10, 11, 12\}$
5. $\{1, 2, 3, 5, 7, 8, 9, 10, 11, 12\}$
6. $\{1, 2, 3, 6, 7, 8, 9, 10, 11, 12\}$
7. $\{1, 2, 3, 4, 5, 6, 7, 10, 11, 12\}$
8. $\{1, 2, 3, 4, 5, 6, 8, 10, 11, 12\}$
9. $\{1, 2, 3, 4, 5, 6, 9, 10, 11, 12\}$
10. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
11. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$
12. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 12\}$

**Theorem 4.4.2:** Let $B_0, B_1$ and $B_2$ denote the association matrices of a partially balanced incomplete block design with two associate classes following group divisible association scheme with $m$ rows each of size $n$. Consider the following structure

$$N^* = \begin{bmatrix} B_2 & J \\ J & B_2 \end{bmatrix}$$

Then $N^*$ becomes the incidence matrix of a singular group divisible type partially balanced incomplete block design with parameters $v^* = 2mn, b^* = 2m, r^* = 2m - 1, k^* = n(2m-1), \lambda_1 = 2m - 1, \lambda_2 = 2(m-1), n_1 = n - 1, n_2 = n(2m-1), m^* = 2m, n^* = n$ after deleting the repeated blocks.

Here $J$ is a $(mn) \times (mn)$ matrix of ones with $m \geq 2, n \geq 2$. 

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**Proof:** Since \( N^* \) denotes the incidence matrix of singular group divisible design containing only zeros and ones, \( N^*N^{*'} \) can be written as:

\[
N^*N^{*'} = \begin{bmatrix}
B_2B_2' + JJ' & B_2J' + JB_2' \\
JB_2' + B_2J' & JJ' + B_2B_2'
\end{bmatrix}
\]

and it can be treated as the treatment structure matrix of a new design D after deleting the repeated blocks in which rows correspond to treatments and columns correspond to blocks. Since the row sum and the column sum of \( N^* \) is \((2m-1) \) and \( n(2m-1) \) respectively, so each treatment in D is replicated \((2m-1)\) times and each block contains \( n(2m-1) \) treatments. Because \( N^*N^{*'} \) is the treatment structure matrix of D, its form shows that treatments can be partitioned into two groups. Taking this partition of treatments as groups of a GD designs, we found that \( \lambda_1 = r^* \), \( \lambda_2 = 2(m-1) \), \( n_1 = n - 1 \), \( n_2 = n(2m-1) \), \( m^* = 2m \), \( n^* = n \). Which clearly shows that \( r^* = \lambda_1 \), hence the design is a singular group divisible design.

**Illustration 4.4.2:** For \( m = 4 \), \( n = 2 \) in the design parameters defined in Theorem 4.4.2, the structure
becomes the incidence matrix of a singular group divisible design with parameters \( v^* = 16 \), \( b^* = 8 \), \( t^* = 7 \), \( k^* = 14 \), \( \lambda_1 = 7 \), \( \lambda_2 = 6 \), \( n_1 = 1 \), \( n_2 = 14 \), \( m^* = 8 \), \( n^* = 2 \), the blocks of which are:

1. \((3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16)\)
2. \((1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16)\)
3. \((1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16)\)
4. \((1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16)\)
5. \((1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16)\)
6. \((1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16)\)
7. \((1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16)\)
8. \((1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14)\)
**Theorem 4.4.3:** Let $B_0$, $B_1$ and $B_2$ denote the association matrices of a partially balanced incomplete block design with two associate classes following group divisible association scheme with $m$ rows each of size $n$. Consider the following structure

$$
N^* = \begin{bmatrix}
B_1 & J \\
J & B_1
\end{bmatrix}
$$

Then under scheme defined in section 4.2.2, $N^*$ becomes the incidence matrix of a symmetric partially balanced incomplete block design with three associate classes with parameters $v^* = b^* = 2mn$, $r^* = k^* = n(m+1)-1$, $\lambda_1 = n(m+1)-2$, $\lambda_2 = mn$, $\lambda_3 = 2(n-1)$, $n_1 = n - 1$, $n_2 = n(m-1)$, $n_3 = mn$.

Here $J$ is a $(mn)\times(mn)$ matrix of ones with $m \geq 2$, $n \geq 2$.

**Proof:** It is easy to verify that the incidence matrix $N^*$ defined above has the required parameters $v^*$, $b^*$, $r^*$ and $k^*$. Under new association scheme defined in section 4.2.2, we found that $n_1 = n - 1$, $n_2 = n(m-1)$ and $n_3 = mn$, so parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$ can be determined easily.

**Illustration 4.4.3:** For $m = 2$, $n = 2$ in the design parameters defined in Theorem 4.4.3, the structure
becomes the incidence matrix partially balanced incomplete block design with three associate classes with parameters \( v^* = b^* = 8 \), \( r^* = k^* = 5 \), \( \lambda_1 = 4 \), \( \lambda_2 = 4 \), \( \lambda_3 = 2 \), \( n_1 = 1 \), \( n_2 = 2 \), \( n_3 = 4 \), following new three associate class association scheme defined in section 4.2.2, the blocks of which are:

1. \((2, 5, 6, 7, 8)\)  
2. \((1, 5, 6, 7, 8)\)  
3. \((4, 5, 6, 7, 8)\)  
4. \((3, 5, 6, 7, 8)\)  
5. \((1, 2, 3, 4, 6)\)  
6. \((1, 2, 3, 4, 5)\)  
7. \((1, 2, 3, 4, 8)\)  
8. \((1, 2, 3, 4, 7)\)

**Theorem 4.4.4:** Let \(B_0\), \(B_1\) and \(B_2\) denote the association matrices of a partially balanced incomplete block design with two associate classes following group divisible association scheme with \( m \) rows each of size \( n \). Consider the following structure

\[
N^* = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
Then under new association scheme defined in section 4.2.2, $N^*$ becomes the incidence matrix of a partially balanced incomplete block design with three associate classes with parameters $v^* = 2mn$, $b^* = 2m$, $r^* = m+1$, $k^* = n(m+1)$, $\lambda_1 = m+1$, $\lambda_2 = m$, $\lambda_3 = 2$, $n_1 = n - 1$, $n_2 = n(m-1)$, $n_3 = mn$ after deleting the repeated blocks.

Here $J$ is a $(mn) \times (mn)$ matrix of ones and $B^2$ is obtained from $B^2$ by interchanging zero’s and one’s with $m \geq 2$, $n \geq 2$.

**Proof:** It is easy to verify that the incidence matrix $N^*$ defined above has the required parameters $v^*$, $b^*$, $r^*$ and $k^*$. Under new association scheme defined in section 4.2.2, we found that $n_1 = n - 1$, $n_2 = n(m-1)$ and $n_3 = mn$, so parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$ can be determined easily.

**Illustration 4.4.4:** For $m = 4$, $n = 2$ in the design parameters defined in Theorem 4.4.4, the structure

\[
N^* = \begin{bmatrix}
1 & 0 & 0 & 0 & . & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & . & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & . & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & . & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & . & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & . & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & . & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & . & 1 & 1 & 1 & 1 \\
. & . & . & . & . & . & . & . & . \\
1 & 1 & 1 & 1 & . & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & . & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & . & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & . & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & . & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & . & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 1
\end{bmatrix}
\]
becomes the incidence matrix of partially balanced incomplete block design with three associate classes with parameters $v^* = 16$, $b^* = 8$, $r^* = 5$, $k^* = 10$, $\lambda_1 = 5$, $\lambda_2 = 4$, $\lambda_3 = 2$, $n_1 = 1$, $n_2 = 6$, $n_3 = 8$, following the new three associate class association scheme defined in section 4.2.2, the blocks of which are:

1. $(1, 2, 9, 10, 11, 12, 13, 14, 15, 16)$
2. $(3, 4, 9, 10, 11, 12, 13, 14, 15, 16)$
3. $(5, 6, 9, 10, 11, 12, 13, 14, 15, 16)$
4. $(7, 8, 9, 10, 11, 12, 13, 14, 15, 16)$
5. $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$
6. $(1, 2, 3, 4, 5, 6, 7, 8, 11, 12)$
7. $(1, 2, 3, 4, 5, 6, 7, 8, 13, 14)$
8. $(1, 2, 3, 4, 5, 6, 7, 8, 15, 16)$

**Theorem 4.4.5:** Let $B_1$ and $B_2$ denote the association matrices of a partially balanced incomplete block design with two associate classes following group divisible association scheme with $m$ rows each of size $n$. Consider

$$\mathcal{N}^* = \begin{bmatrix} B_1 & J & O \\ J & O & B_1 \\ O & B_1 & J \end{bmatrix}$$
Then under new association scheme defined in section 4.2.3, $N^*$ becomes the incidence matrix of a partially balanced incomplete block design with parameters $v^* = b^* = 3mn$, $r^* = k^* = [n(m+1) - 1]$, $\lambda_1 = [n(m + 1) - 2]$, $\lambda_2 = mn$, $\lambda_3 = n - 1$, $n_1 = n - 1$, $n_2 = n(m - 1)$, $n_3 = 2mn$. 

Here $O$ is a $(mn)x(mn)$ Null matrix and $J$ is a $(mn)x(mn)$ matrix of ones with $m \geq 2, n \geq 2$.

**Proof:** It is easy to verify that the incidence matrix $N^*$ defined above has the required parameters $v^*$, $b^*$, $r^*$ and $k^*$. Under new association scheme defined in section 4.2.3, we found that $n_1 = n - 1$, $n_2 = n(m-1)$ and $n_3 = 2mn$, so parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$ can be determined easily.

**Illustration 4.4.5:** For $m = 2$, $n = 2$ in the design parameters defined in Theorem 4.4.5, the structure $N^* = \begin{bmatrix} 0 & 1 & 0 & 0 & . & 1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & . & 1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & . & 1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & . & 1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 0 & . & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 0 & . & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 0 & . & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 0 & . & 0 & 0 & 1 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & 1 & 0 & 0 & . & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & . & 1 & 0 & 0 & 0 & . & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 1 & . & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 1 & 0 & . & 1 & 1 & 1 & 1 \end{bmatrix}$
becomes the incidence matrix of a partially balanced incomplete block design with three associate classes with parameters \( v^* = b^* = 12, \ r^* = k^* = 5, \ \lambda_1 = 4, \ \lambda_2 = 4, \ \lambda_3 = 1, \ n_1 = 1, \ n_2 = 2, \ n_3 = 8 \), following the new three associate class association scheme defined in section 4.2.3, the blocks of which are:

1. (2, 5, 6, 7, 8)  7. (1, 2, 3, 4, 12)
2. (1, 5, 6, 7, 8)  8. (1, 2, 3, 4, 11)
3. (4, 5, 6, 7, 8)  9. (6, 9, 10, 11, 12)
4. (3, 5, 6, 7, 8) 10. (5, 9, 10, 11, 12)
5. (1, 2, 3, 4, 10) 11. (8, 9, 10, 11, 12)
6. (1, 2, 3, 4, 9) 12. (7, 9, 10, 11, 12)

**Remark 4.4.5:** If we consider the following structure

\[
\mathbf{N}^* = \begin{bmatrix}
B_1 & J & J \\
J & J & B_1 \\
J & B_1 & J
\end{bmatrix}
\]

Then \( \mathbf{N}^* \) denotes the incidence matrix of a partially balanced incomplete block design with three associate classes following association scheme defined in section 4.2.3 with parameters \( v^* = b^* = 3mn, \ r^* = k^* = n(2m+1) - 1, \ \lambda_1 = [n(2m + 1) - 2], \ \lambda_2 = 2mn, \ \lambda_3 = [n(m + 2) - 2], \ n_1 = n - 1, \ n_2 = n(m - 1), \ n_3 = 2mn. \)

Here \( J \) is a \((mn)\times(mn)\) matrix of ones with \( m \geq 2, \ n \geq 2. \)
**Illustration 4.4.5:** Taking \( m = 2, n = 2 \) in the design parameters defined in Remark 4.4.5, the structure

\[
N^* = \begin{bmatrix}
0 & 1 & 0 & 0 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 & . & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & . & 1 & 0 & 0 & 0 & . & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & . & 0 & 0 & 0 & 1 & . & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & . & 0 & 0 & 1 & 0 & . & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

becomes the incidence matrix of partially balanced incomplete block design with three associate classes with parameters \( v^* = b^* = 12, \ r^* = k^* = 9, \lambda_1 = 8, \lambda_2 = 8, \lambda_3 = 6, \ n_1 = 1, n_2 = 2, n_3 = 8 \), following new three associate class association scheme defined in section 4.2.3, the blocks of which are:

1. (2, 5, 6, 7, 8, 9, 10, 11, 12)  
2. (1, 5, 6, 7, 8, 9, 10, 11, 12)  
3. (4, 5, 6, 7, 8, 9, 10, 11, 12)  
4. (3, 5, 6, 7, 8, 9, 10, 11, 12)  
5. (1, 2, 3, 4, 6, 9, 10, 11, 12)  
6. (1, 2, 3, 4, 5, 8, 9, 10, 11, 12)  
7. (1, 2, 3, 4, 8, 9, 10, 11, 12)  
8. (1, 2, 3, 4, 7, 9, 10, 11, 12)  
9. (1, 2, 3, 4, 5, 6, 7, 8, 10)  
10. (1, 2, 3, 4, 5, 6, 7, 8, 9)  
11. (1, 2, 3, 4, 5, 6, 7, 8, 12)
Theorem 4.4.6: Let $B_0$, $B_1$ and $B_2$ denote the association matrices of a partially balanced incomplete block design with two associate classes following group divisible association scheme with $m$ rows each of size $n$. Consider the following structure

$$N^* = \begin{bmatrix} B_2 & B_0 \\ B_0 & B_2 \end{bmatrix}$$

Then under new association scheme defined in section 4.2.4, $N^*$ becomes the incidence matrix of a symmetric partially balanced incomplete block design with four associate classes with parameters $v^* = b^* = 2mn$, $r^* = k^* = n(m - 1) + 1$, $\lambda_1 = n(m - 1)$, $\lambda_2 = n(m - 2)$, $\lambda_3 = 0$, $\lambda_4 = 2$, $n_1 = n - 1$, $n_2 = n(m - 1)$, $n_3 = n$, $n_4 = n(m - 1)$ with $m \geq 3$, $n \geq 2$.

**Proof:** It is easy to verify that the incidence matrix $N^*$ defined above has the required parameters $v^*$, $b^*$, $r^*$ and $k^*$. Under new association scheme defined in section 4.2.4, we found that $n_1 = n - 1$, $n_2 = n(m - 1)$, $n_3 = n$, $n_4 = n_2$, so parameters $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ can be determined easily.

**Illustration 4.4.6:** For $m = 2$, $n = 2$ in the design parameter defined in Theorem 4.4.6, the structure
becomes the incidence matrix of partially balanced incomplete block design with four associate classes with parameters $v^* = b^* = 8, \quad r^* = k^* = 3, \quad \lambda_1 = 2, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 2, n_1 = 1, n_2 = 2, n_3 = 2, n_4 = 2$, following new association scheme with four associate classes defined in section 4.2.4, the blocks of which are:

1. (3, 4, 5)  
2. (3, 4, 6)  
3. (1, 2, 7)  
4. (1, 2, 8)  
5. (1, 7, 8)  
6. (2, 7, 8)  
7. (3, 5, 6)  
8. (4, 5, 6)

### 4.5 List of Constructed Designs:

To explain the designs developed in theorems 4.4.1 to 4.4.6, we list some new designs along with their efficiencies corresponding to different values of parameters in following tables:
Table 4.5.1
New Regular Group Divisible Designs Constructed From Theorem 4.4.1 for $r^*, k^* \leq 20; v^*, b^* \leq 25$

<table>
<thead>
<tr>
<th>Design No.</th>
<th>$v^*$</th>
<th>$b^*$</th>
<th>$r^*$</th>
<th>$k^*$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$m^*$</th>
<th>$n^*$</th>
<th>$E$</th>
<th>$E_1$</th>
<th>$E_2$</th>
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<td>12</td>
<td>10</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>4</td>
<td>3</td>
<td>0.9816</td>
<td>0.9900</td>
<td>0.9798</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>16</td>
<td>13</td>
<td>13</td>
<td>12</td>
<td>10</td>
<td>4</td>
<td>4</td>
<td>0.9842</td>
<td>0.9941</td>
<td>0.9818</td>
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<td>16</td>
<td>15</td>
<td>12</td>
<td>4</td>
<td>5</td>
<td>0.9863</td>
<td>0.9961</td>
<td>0.9838</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>18</td>
<td>16</td>
<td>16</td>
<td>15</td>
<td>14</td>
<td>6</td>
<td>3</td>
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<td>0.9961</td>
<td>0.9922</td>
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<tr>
<td>5</td>
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<td>24</td>
<td>19</td>
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<td>6</td>
<td>0.9880</td>
<td>0.9972</td>
<td>0.9855</td>
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</table>

Graph of Average Efficiency Factor (AEF) of Designs Listed in Table-4.5.1 (Theorem 4.4.1)
Table 4.5.2
New Singular GD Designs Corresponding to
Theorem 4.4.2 for \( r^*, k^* \leq 20 \); \( v^*, b^* \leq 25 \)

<table>
<thead>
<tr>
<th>Design No.</th>
<th>( v^* )</th>
<th>( b^* )</th>
<th>( r^* )</th>
<th>( k^* )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( m )</th>
<th>( n )</th>
<th>( E )</th>
<th>( E_1 )</th>
<th>( E_2 )</th>
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<td>10</td>
<td>5</td>
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<td>4</td>
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</table>

Graph of Average Efficiency Factor (AEF) of Designs Listed in Table 4.5.2 (Theorem 4.4.2)
Table 4.5.3  
New PBIB Designs (with Three Associate Classes) Constructed For $r^*, k^* \leq 10$; $v^*, b^* \leq 20$

<table>
<thead>
<tr>
<th>Design No.</th>
<th>$v^*$</th>
<th>$b^*$</th>
<th>$r^*$</th>
<th>$k^*$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$E$</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>Applications of Theorems</th>
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<td>0.9600</td>
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<td>8</td>
<td>7</td>
<td>6</td>
<td>4</td>
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<td>0.9796</td>
<td>0.8396</td>
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<td>6</td>
<td>6</td>
<td>2</td>
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<td>0.9682</td>
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<td>0.9400</td>
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<td>3</td>
<td>2</td>
<td>0.9500</td>
<td>1.0000</td>
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Graph of Average Efficiency Factor (AEF) of Designs Listed in Table 4.5.3 (Theorem 4.4.3)

Graph of Average Efficiency Factor (AEF) of Designs Listed in Table 4.5.3 (Theorem 4.4.4)

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Graph of Average Efficiency Factor (AEF) of Designs Listed in Table 4.5.3 (Theorem 4.4.5)

Graph of Average Efficiency Factor (AEF) of Designs Listed in Table 4.5.3 (Remark 4.4.5)
Table 4.5.4
New PBIB Designs Corresponding to Theorem 4.4.6 (with Four Associate Classes) for $r^*$, $k^* \leq 20$; $v^*$, $b^* \leq 40$

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Graph of Average Efficiency Factor (AEF) of Designs Listed in Table 4.5.4 (Theorem 4.4.6)

From Table 4.5.1, we see that designs 2, 3, 4, and 5 are absolutely new designs except 1 which already exist and are almost equally efficient. We can also see from Table 4.5.2 that designs 3, 4, 5, 7, 8, 9 and 10 except 1, 2 and 6 which already exist are absolutely new designs [c.f. Clatworthy, W.H. (1973)]. The efficiencies of the designs given in table 4.5.3 and table 4.5.4 are fairly good; therefore these designs are of much practical use.

Summary:

Starting from the association matrices of a group divisible partially balanced incomplete block design with two associate classes and following certain juxtaposition pattern involving a type of matrix in which all elements are unity and complement of the association matrices, we have obtained some new partially balanced incomplete block designs with two, three and four associate classes. Some new designs have also been constructed which have fairly good efficiencies and much practical utility. It is interesting to note that the most significant regular group divisible design with r* and k* less than or equal to 10 has been constructed which is very difficult to construct.